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Exhaustive Use of Information on an Equation with Truncated Coefficients

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Abstract—Previously, we proposed algorithms that allow one to find Laurent and regular solutions of linear differential equations with coefficients in the form of truncated formal power series. The solutions contain truncated power series as well. In this paper, we propose some automatic means for confirming the impossibility of obtaining a larger number of terms in these solutions without some additional information on a given equation. The confirmation has the form of a counterexample to the assumption about the possibility of obtaining some additional terms of the solution.

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1. PRELIMINARIES

1.1. Equations and Truncated Series

Suppose that K is an algebraically closed field of characteristic zero. In this paper, we use the common notation for a ring of polynomials in x over K: $K[x]$. A ring of formal power series in x over K is denoted by $K[[x]]$, while a field of formal Laurent series is denoted by $K((x))$. For a nonzero element $a(x) = \sum a_i x^i$ of field $K((x))$, its *valuation* $\mathrm{val}_x a(x)$ is defined by equality $\mathrm{val}_x a(x) = \min\{i \mid a_i \neq 0\}$, with $\mathrm{val}_x 0 = \infty$. Suppose that $t \in \mathbb{Z} \cup \{-\infty\}$; t-truncation $a^{\langle t\rangle}(x)$ is obtained by discarding all terms of $a(x)$ the degrees of which exceed t. Number t is called the *truncation degree*. By $O(x^t)$, where $t \in \mathbb{Z}$, we mean a certain (unspecified) series with a valuation greater than or equal to t. This designation is generally used either when the series is not known or when a particular type of the series is not of interest to us; it is only important that its valuation is not less than t.

In this paper, differential equations are written using operation $\theta = x\dfrac{d}{dx}$ rather than the common differentiation operation $\dfrac{d}{dx}$ (the transition from one form to another is straightforward).

We consider the equations in a truncated form. A *truncated differential equation* is written as follows:

$$\sum_{i=0}^{r} (a_i^{\langle t_i\rangle}(x) + O(x^{t_i+1}))\theta^i y(x) = 0, \tag{1}$$

$a_i^{\langle t_i\rangle}(x) \in K[x]$ and $t_i \geq \deg a_i^{\langle t_i\rangle}(x)$, $i = 0, 1, \ldots, r$, where $O(x^{t_i+1})$ denotes an unspecified part of the series.

We do not know the complete form of the considered equation

$$\begin{aligned} a_r(x)\theta^r y(x) + a_{r-1}(x)\theta^{r-1} y(x) \\ + \cdots + a_0(x) y(x) = 0, \end{aligned} \tag{2}$$

where $a_0(x), \ldots, a_r(x) \in K[[x]]$ and $a_i - a_i^{\langle t_i\rangle}(x) = O(x^{t_i+1})$, $i = 0, \ldots, r$. It is assumed that the leading coefficient $a_r(x)$ is not zero and the valuation of at least one of the series $a_0(x), \ldots, a_r(x)$ is zero.

1.2. Laurent and Regular Solutions

A solution of differential equation (2) that is a formal Laurent series is called a *Laurent solution*.

A *regular solution* has the form

$$y(x) = x^{\lambda} w(x),$$

where $\lambda \in K$ and $w(x) \in K((x))[\ln x]$. Each such solution is written as

$$x^{\lambda} \sum_{s=0}^{m} w_s(x) \ln^s x, \tag{3}$$

where $m \in \mathbb{Z}_{\geq 0}$ and $w_s(x) \in K((x))$, $s = 0, 1, \ldots, m$.

2. PROBLEM STATEMENT

Truncated (or shortened) series in the role of coefficients of equations of various types are of both theoretical and practical interest; these series become the subject of various studies (see, for example, [1, 2]). In [3–7], we considered linear ordinary differential equations with coefficients in the form of truncated power series. We discussed the question of what can be learned from the equations represented in this way about their Laurent and regular solutions. We were interested in finding the maximum possible number of coefficients that are invariant to various possible prolongations of a given truncated equation. The *prolongation* of a truncated series is a series (possibly, also truncated) the initial terms of which coincide with known initial terms of the original truncated series. In turn, the prolongation of a truncated equation is an equation the coefficients of which are prolongations of coefficients of the original equation.

For example, let us consider the equation

$$\begin{gathered}(-1 + x + O(x^3))\theta^2 y(x) \\ + (-2 + x + O(x^3))\theta y(x) + O(x^3) y(x) = 0.\end{gathered} \tag{4}$$

Using the algorithm from [3, 4], we find that any equation of form (2), i.e., an equation with completely specified coefficients that is a prolongation of (4), has Laurent solutions with valuations –2 and 0. Moreover, for any prolongation of the equation, any Laurent solution with valuation –2 is a prolongation of the truncated series

$$\frac{\_c_1}{x^2} - \frac{2\_c_1}{x} + \_c_2 + O(x) \tag{5}$$

for some $\_c_1, \_c_2 \in K$, where $\_c_1 \neq 0$. Also, for any prolongation of (4), any of its Laurent solutions with zero valuation is a prolongation of the truncated series

$$\_c_2 + O(x^3) \tag{6}$$

for some $\_c_2 \in K$, where $\_c_2 \neq 0$. Expressions (5) and (6) are called *truncated Laurent solutions* of equation (4). The algorithm from [3, 4] allows one to construct truncated Laurent solutions of the maximum possible truncation degree. In the case of (4), for solutions with valuation –2, the maximum possible truncation degree is 0; for solutions with zero valuation, the degree is 2. Hereinafter, by truncated solutions, we mean solutions with the maximum possible truncation degree.

For example, two different prolongations of equation (4) are

$$\begin{gathered}(-1 + x - \mathbf{x^3} + \mathbf{O(x^4)})\theta^2 y(x) \\ + (-2 + x - \mathbf{x^3} + \mathbf{O(x^4)})\theta y(x) \\ + (-\mathbf{x^3} + \mathbf{O(x^4)}) y(x) = 0\end{gathered} \tag{7}$$

and

$$\begin{gathered}(-1 + x + \mathbf{x^3} + \mathbf{O(x^4)})\theta^2 y(x) \\ + (-2 + x + \mathbf{x^3} + \mathbf{O(x^4)})\theta y(x) \\ + (\mathbf{x^3} + \mathbf{O(x^4)}) y(x) = 0\end{gathered} \tag{8}$$

(hereinafter, additional terms of prolongations are highlighted in bold). Truncated Laurent solutions of equation (7) are

$$\frac{\_c_1}{x^2} - \frac{2\_c_1}{x} + \_c_2 - \mathbf{\_c_1 x} + \mathbf{O(x^2)},$$

$$\_c_2 - \frac{\mathbf{\_c_2 x^3}}{\mathbf{15}} + \mathbf{O(x^4)}.$$

Truncated Laurent solutions of equation (8) are

$$\frac{\_c_1}{x^2} - \frac{2\_c_1}{x} + \_c_2 + \mathbf{\_c_1 x} + \mathbf{O(x^2)},$$

$$\_c_2 + \frac{\mathbf{\_c_2 x^3}}{\mathbf{15}} + \mathbf{O(x^4)}.$$

It can be seen that (5) and (6) are truncated solutions of equation (4) with the maximum possible truncation degree: solutions of (7) and (8) show that the coefficients of the additional terms of the solutions depend on the additional terms in the coefficients of the prolongations of the original equation.

The algorithms for constructing truncated Laurent and regular solutions were described in the papers mentioned above. It means that these algorithms provide the exhaustive use of information about a given equation. The algorithms were implemented in the Maple computer algebra system [8].

In this paper, we investigate an automatic confirmation of this exhaustive use of information, i.e., a confirmation that it is impossible to add additional terms to the constructed truncated solutions while preserving their invariance to all prolongations of a given equation. For this purpose, it is sufficient to present a counterexample with two different prolongations of the given equation that cause the occurrence of different additional terms in its solutions.

The preliminary results of this work were published in [9].

3. SOLUTION METHOD

The procedures are based on finding Laurent and regular solutions with *literals*, i.e., symbols used to represent unspecified coefficients of a series involved in the equations (see [5]). These symbols denote the coefficients of the terms the degrees of which exceed the truncation degree of the series. Finding solutions with the use of literals means representing the subsequent (non-invariant to all possible prolongations) terms of the series involved in the solution as formulas that contain literals, i.e., expressing them in terms of unspecified coefficients. This allows us to clarify the influence of unspecified coefficients on the subsequent terms of the series in the solution.

Solutions of (4) with literals have the following form:

$$\frac{\_c_1}{x^2} - \frac{2\_c_1}{x} + \_c_2 + \left(\frac{1}{3}U_{0,3} - \frac{2}{3}U_{1,3} + \frac{4}{3}U_{2,3}\right)\_c_1 x + O(x^2),$$

$$\_c_2 + \frac{U_{0,3}}{15}\_c_2 x^3 + O(x^4), \tag{9}$$

where symbols $U_{i,j}$ are literals that correspond to unspecified coefficients of x^j in the power series that is a coefficient of $\theta^i y(x)$ in the equation. In the general case, solution coefficients that are non-invariant prolongations of the constructed truncated solutions are represented as polynomials in literals.

Lemma 1. *For any* $p_i(x_1, \ldots, x_l) \in K[x_1, \ldots, x_l]\backslash K$ $(i = 1, \ldots, m)$, *there are* $\alpha_1, \ldots, \alpha_l, \beta_1, \ldots, \beta_l \in K$ *for which* $p_i(\alpha_1, \ldots, \alpha_l) \neq p_i(\beta_1, \ldots, \beta_l)$ $(i = 1, \ldots, m)$.

Proof. First, by induction on n, we show that, for any polynomial $P(x_1, \ldots, x_n) \in K[x_1, \ldots, x_n]\backslash K$ $(n \geq 1)$, there are $\alpha_1, \ldots, \alpha_n \in K$ for which $P(\alpha_1, \ldots, \alpha_n) \neq 0$. For $n = 1$, the statement is obvious because $P(x_1)$ has a finite number of roots and field K is infinite. Suppose that $n > 1$ and the statement holds for $1, \ldots, n-1$. We write $P(x_1, \ldots, x_n)$ as a polynomial in x_n and assume that $q(x_1, \ldots, x_{n-1})$ is a nonzero coefficient of this polynomial. By the induction hypothesis, there are $\alpha_1, \ldots, \alpha_{n-1} \in K$ such that $q(\alpha_1, \ldots, \alpha_{n-1}) \neq 0$. Hence, $P(\alpha_1, \ldots, \alpha_{n-1}, x_n)$ is a nonzero polynomial in x_n. According to the induction base case (the case of one variable), a suitable α_n exists.

Let us now consider nonzero polynomial $P(x_1, \ldots, x_l, x_{l+1}, \ldots, x_{2l})$ that is equal to a product of polynomials $p_i(x_1, \ldots, x_l) - p_i(x_{l+1}, \ldots, x_{2l})$, $i = 1, \ldots, m$ (polynomial $p_i(x_{l+1}, \ldots, x_{2l})$ is obtained by simply replacing variables $x_1, \ldots, x_l$ with new variables $x_{l+1}, \ldots, x_{2l}$). Based on what has been proved above, there are $\alpha_1, \ldots, \alpha_{2l} \in K$ such that $P(\alpha_1, \ldots, \alpha_{2l}) \neq 0$. We can leave $\alpha_1, \ldots, \alpha_l$ unchanged and assume that $\beta_1 = \alpha_{l+1}, \ldots, \beta_l = \alpha_{2l}$. □

From Lemma 1, we obtain the following theorem, which substantiates the algorithm for confirming the exhaustive use of information contained in the equation with truncated coefficients. The algorithm itself relies on the construction of solutions with literals.

Theorem 1. *Suppose that solutions of equation* (1) *involve m truncated power series* $c_{i0} + c_{i1}x + \ldots + c_{ik_i}x^{k_i} + p_i(u_1, \ldots, u_l)x^{k_i+1} + O(x^{k_i+2})$, $1 \leq i \leq m$, *where* $u_1, \ldots, u_l$ *are literals* (*unspecified coefficients of power series-coefficients of the equation*). *Then, there are* $\alpha_1, \ldots, \alpha_l, \beta_1, \ldots, \beta_l \in K$ *such that two different prolongations of the equation that correspond to substitutions* $u_j = \alpha_j$ *and* $u_j = \beta_j$ $(j = 1, \ldots, l)$ *lead to the occurrence of different additional terms in the truncated series involved in the solutions, which confirms the exhaustive use of information contained in the truncated coefficients of equation* (1).

For the solutions of (4) with literals (9), the first non-invariant coefficients of the series are given by the following expressions:

$$\left(\frac{1}{3}U_{0,3} - \frac{2}{3}U_{1,3} + \frac{4}{3}U_{2,3}\right)\_c_1, \tag{10}$$

$$\frac{U_{0,3}}{15}\_c_2. \tag{11}$$

In this case, the solutions involve two truncated series, and we need to find different values of all literals $U_{0,3}$, $U_{1,3}$, and $U_{2,3}$ such that the values of each expression (10) and (11) do not coincide for these different values of the literals. According to Theorem 1, these values exist. In particular, $U_{0,3} = -1$, $U_{1,3} = -1$, and $U_{2,3} = -1$ correspond to equation (7), while $U_{0,3} = 1$, $U_{1,3} = 1$, and $U_{2,3} = 1$ correspond to equation (8). It should be noted that the choice of $U_{0,3} = -1$, $U_{1,3} = 1$, $U_{2,3} = 1$ and $U_{0,3} = 1$, $U_{1,3} = -2$, $U_{2,3} = -1$ also provides different prolongations of (4) that, however, have the same solution prolongations with valuation -2, because expression (10) has the same value of $\frac{1}{3}$ for these two different pairs of literal values. For these two different pairs of values, expression (11) has different values; hence, the prolongations of the solution for zero valuation are different. Nevertheless, these pairs of different values do not constitute a counterexample, because it is necessary that all series involved in the solutions have different prolongations; otherwise, the exhaustive use of information contained in the equation remains unconfirmed.

4. THE CASE OF VARIATION IN THE DEGREE OF $\ln x$ IN SOLUTIONS

The algorithms for finding truncated solutions considered in [3–7] construct only those truncated solutions for which the degree of occurrence of $\ln x$ is invariant. For example, any prolongation with completely specified coefficients of the equation

$$(-1 + x + O(x^2))\theta^2 y(x) + (-2 + x^2 + O(x^3))\theta y(x) + O(x^4)y(x) = 0 \tag{12}$$

has regular solutions with valuations −2 and 0. However, for some prolongations of (12), all regular solutions are Laurent ones, whereas for its other prolongations, regular solutions with valuation −2 have form (3) with $m = 1$ and $w_1(x) \neq 0$; i.e., they are not Laurent solutions.

According to [6, Remark 5], when computing truncated solutions, a finite set of polynomials in literals is formed. Each polynomial P from this set has coefficients in the form of linear combinations over K of arbitrary constants $\_c_1, \ldots, \_c_r$, which are introduced when constructing truncated solutions. For instance, for equation (12), we obtain

$$P = 4U_{2,2}\_c_1 - 6\_c_1.$$

According to the algorithm for constructing regular solutions from [6], the values of the literals for which these polynomials identically vanish define equation prolongations such that the degree of occurrence of $\ln x$ in regular solutions for these prolongations is lower than that for the prolongations for which some P do not identically vanish.

Automatic provision of these prolongations also confirms the exhaustive use of information about a given truncated equation. It should be noted that not every truncated equation has prolongations that differ in the degree of occurrence of $\ln x$ in regular solutions. Counterexamples for equation (12) and their truncated solutions are considered in Section 5.3 (see Example 6).

5. IMPLEMENTATION AND EXAMPLES

Below, we describe the implemented procedures for finding counterexample prolongations. The exhaustive use of information about an equation with truncated power series coefficients in truncated solutions is confirmed using the `ExhaustiveUseConfirmation` procedure implemented in the Maple computer algebra system [8]. We also implemented auxiliary procedures `DifferentProlongationExtras`, `ConstructProlongation`, and `DifferentLnDegreeExtras`, which are also considered in this section. The procedures are built into the `TruncatedSeries` package [10–12], which was used to implement our algorithms from [3–7].

5.1. Laurent Solutions

Example 1. Let us consider the following equation with coefficients in the form of truncated series and construct its Laurent solution using the `TruncatedSeries` package.

```
> eq := (-1+x+x^2+O(x^3))*theta(y(x),x,2)+
        (-2+O(x^3))*theta(y(x),x,1)+
        (x+6*x^2+O(x^4))*y(x);
```

$$eq := (-1 + x + x^2 + O(x^3))\theta(y(x), x, 2) + (-2 + O(x^3))\theta(y(x), x, 1) + (x + 6x^2 + O(x^4))y(x)$$

```
> sol := TruncatedSeries:-
      LaurentSolution(eq,y(x));
```

$$sol := \left[\frac{\_c_1}{x^2} - \frac{5\_c_1}{x} + \_c_2 + O(x),\ \_c_2 + \frac{x\_c_2}{3} + \frac{5x^2\_c_2}{6} + \frac{13x^3\_c_2}{30} + O(x^4)\right].$$

The execution of the `ExhaustiveUseConfirmation` procedure (the option `'laurent'` is used to specify that the result for Laurent solutions is required) confirms the exhaustive use of information about the equation by presenting two different equation prolongations that lead to two different solution prolongations. The procedure prints text comments with details about these two different prolongations ("Equation prolongation #1" and "Equation prolongation #2") and about their solutions ("The equation solution"). It is shown that the presented equation prolongations have different additional terms ("Additional term(s) in the equation prolongation") and that the solutions of both the prolongations are different solution prolongations of the given equation with different additional terms in these solutions ("Additional term(s) in the equation solution").

```
> TruncatedSeries:-
    ExhaustiveUseConfirmation(sol,eq,y(x),
                              'laurent');
  Equation prolongation #1
```

$$(-1 + x + x^2 - x^3 + O(x^4))\theta(y(x), x, 2) + (-2 - x^3 + O(x^4))\theta(y(x), x, 1) + (x + 6x^2 - x^4 + O(x^5))y(x).$$

Additional term(s) in the equation prolongation:

$$y(x)(-x^4 + O(x^5)) + \theta(y(x), x, 1)(-x^3 + O(x^4)) + \theta(y(x), x, 2)(-x^3 + O(x^4)).$$

The equation solution:

$$\left[\frac{\_c_1}{x^2} - \frac{5\_c_1}{x} + \_c_2 + x\left(\frac{\_c_2}{3} - \frac{37\_c_1}{3}\right) + O(x^2),\right.$$

$$\_c_2 + \frac{x\_c_2}{3} + \frac{5x^2\_c_2}{6} + \frac{13x^3\_c_2}{30} + \frac{11x^4\_c_2}{24} + O(x^5)\Bigg].$$

Additional term(s) in the equation solution:

$$\left[x\left(\frac{\_c_2}{3} - \frac{37\_c_1}{3}\right) + O(x^2), \frac{11x^4\_c_2}{24} + O(x^5)\right].$$

```
Equation prolongation #2
```

$$(-1 + x + x^2 + x^3 + O(x^4))\theta(y(x), x, 2)$$
$$+(-2 + x^3 + O(x^4))\theta(y(x), x, 1)$$
$$+(x + 6x^2 + x^4 + O(x^5))y(x).$$

Additional term(s) in the equation prolongation:

$$y(x)()x^4 + O(x^5) + \theta(y(x), x, 1)(x^3 + O(x^4))$$
$$+ \theta(y(x), x, 2)(x^3 + O(x^4)).$$

The equation solution:

$$\Bigg[\frac{\_c_1}{x^2} - \frac{5\_c_1}{x} + \_c_2 + x\left(\frac{\_c_2}{3} - 11\_c_1\right) + O(x^2),$$
$$\_c_2 + \frac{x\_c_2}{3} + \frac{5x^2\_c_2}{6} + \frac{13x^3\_c_2}{30} + \frac{43x^4\_c_2}{72} + O(x^5)\Bigg].$$

Additional term(s) in the equation solution:

$$\left[x\left(\frac{\_c_2}{3} - 11\_c_1\right) + O(x^2), \frac{43x^4\_c_2}{72} + O(x^5)\right].$$

Example 2. Let us consider another prolongation of this equation with other additional terms (the auxiliary procedure `ConstructProlongation` is used) and construct its Laurent solutions.

```
> eq1 := TruncatedSeries:-
        ConstructProlongation(
        theta(y(x),x,1)*x^3,eq,y(x));
```

$$eq1 := (-1 + x + x^2 + O(x^3))\theta(y(x), x, 2)$$
$$+ (-2 + x^3 + O(x^4))\theta(y(x), x, 1)$$
$$+ (x + 6x^2 + O(x^4))y(x)$$

```
> TruncatedSeries:-
  LaurentSolution(eq1,y(x));
```

$$\Bigg[\frac{\_c_1}{x^2} - \frac{5\_c_1}{x} + \_c_2 + O(x),$$
$$\_c_2 + \frac{x\_c_2}{3} + \frac{5x^2\_c_2}{6} + \frac{13x^3\_c_2}{30} + O(x^4)\Bigg].$$

The solution coincides with the solution of equation `eq`. This implies that, to confirm the exhaustive use of information about the equation, it is not sufficient to simply construct solutions of two different random prolongations. The information provided by additional terms in a random prolongation does not necessarily cause the occurrence of any additional terms in solutions of the equation; hence, this prolongation cannot be used as a counterexample.

Example 3. Let us consider the following equation and construct its Laurent solutions.

```
> eq := (x + O(x^2))*theta(y(x), x, 1) +
      O(x^2)*y(x);
```

$$eq := (x + O(x^2))\theta(y(x), x, 1) + O(x^2)y(x)$$

```
> sol := TruncatedSeries:-
         LaurentSolution(eq,y(x));
```

$$sol := [\_c_1 + O(x)].$$

Instead of using the `ExhaustiveUseConfirmation` procedure, the exhaustive use of information about the equation can be confirmed step by step using two auxiliary procedures. This method is more preferable than the use of the text comments printed by the `ExhaustiveUseConfirmation` procedure in the cases where, e.g., the details of the counterexample are required in some subsequent algorithmic processing.

At the first step, the `DifferentProlongationExtras` procedure (again, the `'laurent'` option is used to specify that the result for Laurent solutions is required) provides two different variations of additional terms to construct two different prolongations of the equation:

```
> dp := TruncatedSeries:-
        DifferentProlongationExtras
(eq,y(x),'laurent');
```

$$dp := [y(x)(-x^2 + O(x^3)), y(x)(x^2 + O(x^3))].$$

At the second step, the `ConstructProlongation` procedure is executed twice to construct these two different prolongations of the equation:

```
> eq1 := TruncatedSeries:-
         ConstructProlongation
(dp [1],eq,y(x));
```

$$eq1 := (x + O(x^2))\theta(y(x), x, 1) + y(x)(-x^2 + O(x^3))$$

```
> eq2 := TruncatedSeries:-
        ConstructProlongation
(dp[2],eq,y(x));
```

$$eq2 := (x + O(x^2))\theta(y(x), x, 1) + y(x)(x^2 + O(x^3)).$$

Finally, at the third step, Laurent solutions are constructed for each prolongation:

```
> sol1 := TruncatedSeries:-
         LaurentSolution(eq1,y(x));
```

$$sol1 := [\_c_1 + x\_c_1 + O(x^2)]$$

```
> sol2 := TruncatedSeries:-
          LaurentSolution(eq2,y(x));
```

$$sol2 := [\_c_1 - x\_c_1 + O(x^2)].$$

The different prolongations of the equation lead to two different prolongations of the solution.

5.2. Regular Solutions

Example 4. Let us consider the following equation and construct its regular solutions.

```
> eq := (-1+x+x^2+O(x^3))*theta(y(x), x, 2)+
(-2 + x^2 + O(x^3))*theta(y(x),x,1)  +
O(x^4)*y(x)
```

$$eq := (-1 + x + x^2 + O(x^3))\theta(y(x), x, 2) + (-2 + x^2 + O(x^3))\theta(y(x), x, 1) + O(x^4)y(x)$$

```
> sol := TruncatedSeries:-
         RegularSolution(eq,y(x));
```

$$sol := \left[-\frac{\_c_1}{x^2} + \frac{4\_c_1}{x} + \_c_2 + O(x) + \ln(x)(\_c_1 + O(x^4)), \_c_2 + O(x^4)\right].$$

We carry out the step-by-step confirmation of the exhaustive use of information about this equation as in Example 3.

The execution of the `DifferentProlongation-Extras` procedure (in this case, the `'regular'` option is used to specify that the result for regular solutions is required) provides two different variations of additional terms to construct two different prolongations of the equation:

```
> dp := TruncatedSeries:-
        DifferentProlongationExtras(
eq,y(x), 'regular');
```

$$dp := [y(x)(-x^4 + O(x^5)) + \theta(y(x), x, 2)(-x^3 + O(x^4)) + \theta(y(x), x, 1)(-x^3 + O(x^4)),$$
$$y(x)(x^4 + O(x^5)) + \theta(y(x), x, 2)(x^3 + O(x^4)) + \theta(y(x), x, 1)(x^3 + O(x^4))].$$

The `ConstructProlongation` procedure is used to construct these two prolongations:

```
> eq1 := TruncatedSeries:-
          ConstructProlongation
(dp [1],eq,y(x));
```

$$eq1 := (-1 + x + x^2 - x^3 + O(x^4))\theta(y(x), x, 2) + (-2 + x^2 - x^3 + O(x^4))\theta(y(x), x, 1) + (-x^4 + O(x^5))y(x)$$

```
> eq2 := TruncatedSeries:-
         ConstructProlongation
(dp[2],eq,y(x));
```

$$eq2 := (-1 + x + x^2 + x^3 + O(x^4))\theta(y(x), x, 2) + (-2 + x^2 + x^3 + O(x^4))\theta(y(x), x, 1) + (x^4 + O(x^5))y(x).$$

Then, regular solutions of each prolongation are constructed:

```
> sol1 := TruncatedSeries:-
          RegularSolution(eq1,y(x));
```

$$sol1 := \left[-\frac{\_c_1}{x^2} + \frac{4\_c_1}{x} + \_c_2 + \frac{2\_c_1 x}{3} + O(x^2) + \ln(x)\left(\_c_1 - \frac{x^4\_c_1}{24} + O(x^5)\right), \_c_2 - \frac{x^4\_c_1}{24} + O(x^5)\right]$$

```
> sol2 := TruncatedSeries:-
          RegularSolution(eq2,y(x));
```

$$sol2 := \left[-\frac{\_c_1}{x^2} + \frac{4\_c_1}{x} + \_c_2 - \frac{2\_c_1 x}{3} + O(x^2) + \ln(x)\left(\_c_1 + \frac{x^4\_c_1}{24} + O(x^5)\right), \_c_2 + \frac{x^4\_c_1}{24} + O(x^5)\right].$$

The different prolongations of the equation lead to two different prolongations of the solution.

Example 5. Let us consider another equation and construct its regular solutions.

```
>eq := (1+x^2+O(x^3))*theta(y(x),x,3)+
        (4-x+1/2*x^2+O(x^3))*
theta(y(x),x,2)+(4-2*x+x^2+O(x^3))
*theta(y(x),x,1)+O(x^3)*y(x);
```

$$eq := (x^2 + 1 + O(x^3))\theta(y(x), x, 3) + \left(4 - x + \frac{x^2}{2} + O(x^3)\right)\theta(y(x), x, 2)$$
$$+(x^2 - 2x + 4 + O(x^3))\theta(y(x), x, 1) + O(x^3)y(x)$$

```
> sol := TruncatedSeries:-
          RegularSolution(eq,y(x));
```

$$sol := \left[\frac{\frac{21\_c_1}{16} + \frac{\_c_2}{2}}{x^2} + \frac{\_c_1}{x} + \_c_3 + O(x)\right.$$
$$+\ln(x)\left(\frac{\_c_1}{2x^2} + \_c_2 + O(x)\right) + \ln(x)^2\left(\frac{\_c_1}{2} + O(x^3)\right),$$

$$\left.\frac{\_c_2}{2x^2} + \_c_3 + O(x) + \ln(x)(\_c_2 + O(x^3)), \_c_3 + O(x^3)\right].$$

The execution of the `ExhaustiveUseConfirmation` procedure (again, the `'regular'` option is used to obtain the result for regular solutions) confirms the exhaustive use of information about this equation. The procedure prints the text comments as in Example 1.

```
> TruncatedSeries:-
  ExhaustiveUseConfirmation(sol,eq,y(x),
                                     'regular');
  Equation prolongation #1
```

$$(x^2 + 1 - x^3 + O(x^4))\theta(y(x), x, 3)$$
$$+\left(4 - x + \frac{x^2}{2} - x^3 + O(x^4)\right)\theta(y(x), x, 2)$$
$$+(x^2 - 2x + 4 - x^3 + O(x^4))\theta(y(x), x, 1)$$
$$+ (-x^3 + O(x^4))y(x).$$

Additional term(s) in the equation prolongation:

$$\theta(y(x), x, 3)(-x^3 + O(x^4))$$
$$+ \theta(y(x), x, 2)(-x^3 + O(x^4))$$
$$+\theta(y(x), x, 1)(-x^3 + O(x^4)) + y(x)(-x^3 + O(x^4)).$$

The equation solution:

$$\left[\frac{\frac{21\_c_1}{16} + \frac{\_c_2}{2}}{x^2} + \frac{\_c_1}{x} + \_c_3 + \frac{61\_c_1 x}{432} - \frac{\_c_2 x}{18} + O(x^2)\right.$$
$$+\ln(x)\left(\frac{\_c_1}{2x^2} + \_c_2 - \frac{\_c_1 x}{18} + O(x^2)\right)$$
$$+ \ln(x)^2\left(\frac{\_c_1}{2} + x^3\frac{\_c_1}{150} + O(x^4)\right),$$
$$\frac{\_c_2}{2x^2} + \_c_3 - \frac{\_c_2 x}{18} + O(x^2)$$
$$+\ln(x)\left(\_c_2 + \frac{x^3\_c_2}{75} + O(x^4)\right),$$
$$\left.\_c_3 + \frac{x^3\_c_3}{75} + O(x^4)\right].$$

Additional term(s) in the equation solution:

$$\left[\left[\left(x^3\frac{\_c_1}{150} + O(x^4)\right)\ln(x)^2 + \left(-\frac{\_c_1 x}{18} + O(x^2)\right)\ln(x)\right.\right.$$
$$+ \frac{61\_c_1 x}{432} - \frac{\_c_2 x}{18} + O(x^2),$$
$$\left(\frac{x^3\_c_2}{75} + O(x^4)\right)\ln(x) - \frac{\_c_2 x}{18} + O(x^2),$$
$$\left.\frac{x^3\_c_3}{75} + O(x^4)\right].$$

```
Equation prolongation #2
```

$$(x^2 + 1 + x^3 + O(x^4))\theta(y(x), x, 3)$$
$$+\left(4 - x + \frac{x^2}{2} + x^3 + O(x^4)\right)\theta(y(x), x, 2)$$
$$+ (x^2 - 2x + 4 + x^3 + O(x^4))\theta(y(x), x, 1)$$
$$+ (x^3 + O(x^4))y(x).$$

Additional term(s) in the equation prolongation:

$$\theta(y(x), x, 3)(x^3 + O(x^4)) + \theta(y(x), x, 2)(x^3 + O(x^4))$$
$$+ \theta(y(x), x, 1)(x^3 + O(x^4)) + y(x)(x^3 + O(x^4)).$$

The equation solution:

$$\left[\frac{\frac{21\_c_1}{16} + \frac{\_c_2}{2}}{x^2} + \frac{\_c_1}{x}\right.$$
$$+ \_c_3 - \frac{\left(\frac{1175\_c_1}{24} - 75\_c_2\right)x}{150} + O(x^2)$$
$$+ \ln(x)\left(\frac{\_c_1}{2x^2} + \_c_2 + \frac{\_c_1 x}{2} + O(x^2)\right)$$
$$+ \ln(x)^2\left(\frac{\_c_1}{2} - \frac{x^3\_c_1}{150} + O(x^4)\right),$$
$$\frac{\_c_2}{2x^2} + \_c_3 + \frac{\_c_2 x}{2} + O(x^2)$$
$$+\ln(x)\left(\_c_2 - \frac{x^3\_c_2}{75} + O(x^4)\right),$$
$$\left.\_c_3 - \frac{x^3\_c_3}{75} + O(x^4)\right].$$

Additional term(s) in the equation solution:

$$\left[\left[\left(-\frac{x^3\_c_1}{150} + O(x^4)\right)\ln(x)^2 + \left(O(x^2) + \frac{\_c_1 x}{2}\right)\ln(x)\right.\right.$$
$$- \frac{\left(\frac{1175\_c_1}{24} - 75\_c_2\right)x}{150} + O(x^2),$$

$$\left(-\frac{x^3\_c_2}{75}+O(x^4)\right)\ln(x)+\frac{\_c_2 x}{2}+O(x^2),$$

$$\left.-\frac{x^3\_c_3}{75}+O(x^4)\right].$$

5.3. Variation in the Degree of Occurrence of $\ln x$ in the Solution

Example 6. Let us construct regular solutions of the following equation.

```
> eq := (-1+x+O(x^2))*theta(y(x), x, 2)+
       (-2 + x^2 + O(x^3))*theta(y(x),x,1)+
       O(x^4)*y(x)
```

$$eq := (-1+x+O(x^2))\theta(y(x),x,2)$$

$$+(-2+x^2+O(x^3))\theta(y(x),x,1)+O(x^4)y(x)$$

```
> sol := TruncatedSeries:-
         RegularSolution(eq,y(x));
```

$$sol := [\_c_1+O(x^4)].$$

It can be seen that, in this case, the truncated regular solutions are constructed as Laurent solutions with zero valuation. Let us compute the prolongations that have regular solutions with different degrees of occurrence of $\ln x$ by using the `DifferentLnDegreeExtras` procedure.

```
> dp := TruncatedSeries:-
        DifferentLnDegreeExtras(eq,y(x));
```

$$dp := \left[\theta(y(x),x,2)\left(\frac{3}{2}x^2+O(x^3)\right),\right.$$

$$\left.\theta(y(x),x,2)(2x^2+O(x^3))\right]$$

The ConstructProlongation procedure is used to construct these two different prolongations:

```
> eq1 := TruncatedSeries:-
         ConstructProlongation
(dp [1],eq,y(x));
```

$$eq1 := \left(-1+x+\frac{3}{2}x^2+O(x^3)\right)\theta(y(x),x,2)$$

$$+(-2+x^2+O(x^3))\theta(y(x),x,1)+O(x^4)y(x)$$

```
> eq2 := TruncatedSeries:-
         ConstructProlongation
(dp [2],eq,y(x));
```

$$eq2 := (-1+x+2x^2+O(x^3))\theta(y(x),x,2)$$

$$+(-2+x^2+O(x^3))\theta(y(x),x,1)+O(x^4)y(x).$$

Then, regular solutions of each prolongation are constructed:

```
> sol1 := TruncatedSeries:-
          RegularSolution(eq1,y(x));
```

$$sol1 := \left[\frac{\_c_1}{x^2}-\frac{4\_c_1}{x}+\_c_2+O(x),\_c_2+O(x^4)\right]$$

```
> sol2 := TruncatedSeries:-
          RegularSolution(eq2,y(x));
```

$$sol2 := \left[\frac{\_c_1}{x^2}-\frac{4\_c_1}{x}+\_c_2+O(x)\right.$$

$$\left.+\ln(x)(\_c_1+O(x^4)),\_c_2+O(x^4)\right].$$

For the first prolongation of the equation, all solutions are Laurent ones; in addition to Laurent solutions with zero valuation, Laurent solutions with valuation –2 are constructed. For the second prolongation, the solutions with valuation –2 are regular ones and involve $\ln x$.

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