Journal of Symbolic Computation 127 (2025) 102350

Contents lists available at ScienceDirect

Journal of Symbolic Computation

journal homepage: www.elsevier.com/locate/jsc

On the dimension of the solution space of linear difference equations over the ring of infinite sequences

Sergei Abramov [a], Gleb Pogudin [b]

[a] *Federal Research Center "Computer Science and Control" of the Russian Academy of Sciences, Moscow, Russia*
[b] *LIX, CNRS, Ecole polytechnique, Institute Polytechnique de Paris, Paris, France*

A R T I C L E I N F O

Article history:
Received 7 December 2023
Received in revised form 3 May 2024
Accepted 20 June 2024
Available online 27 June 2024

In memory of Marko Petkovšek

Keywords:
Linear difference operator with sequence coefficients
Solution space dimension
Undecidability

A B S T R A C T

For a linear difference equation with the coefficients being computable sequences, we establish algorithmic undecidability of the problem of determining the dimension of the solution space including the case when some additional prior information on the dimension is available.

In memory of Marko Petkovšek

1. Introduction

The present paper studies equations whose coefficients and solutions of interest are two-sided infinite sequences. Infinite sequences are used in many areas of mathematics. When working with these sequences the way they are represented plays an important role. In this article, an algorithmic approach is used: the sequence is defined by an algorithm (each sequence has its own) for calculating the value of an element by the index of this element. More formally, we will call a two-sided sequence of rational numbers $\{v(n)\}_{n\in\mathbb{Z}}$ *computable* if it is given by an algorithm computing the value of $v(n)$ for any given $n \in \mathbb{Z}$. Other approaches are also possible, for example, if coefficient sequences satisfy

E-mail addresses: sergeyabramov@mail.ru (S. Abramov), gleb.pogudin@polytechnique.edu (G. Pogudin).

https://doi.org/10.1016/j.jsc.2024.102350

linear recurrences with constant coefficients, we naturally come to the concept of C^2-finite sequences considered, for example, by Jiménez-Pastor et al. (2023).

Throughout the paper, R stands for the ring of two-sided sequences having rational number terms with respect to termwise addition and multiplication. For a linear difference equation

$$a_r(n)y(n+r)+\cdots+a_1(n)y(n+1)+a_0(n)y(n)=0 \tag{1}$$

with computable $a_r(n),\ldots,a_0(n)\in R$ as coefficients, we consider the $\mathbb{Q}$-vector space of solutions in R. In the sequel, “equation” is always understood as an equation of the form (1), and “solution” is a solution belonging to R.

In this paper, we establish algorithmic undecidability of the problem of computing the dimension of the solution space of an equation of the form (1) and the problem of testing whether an equation has nonzero solutions at all (Section 3). Furthermore, it is proven that even when a finite set of possible values of the dimension of the solution space is known in advance, if this set contains more than one element, then in the general case the dimension cannot be found algorithmically (Section 4). On the other hand, we note the existence of naturally arising problems admitting an algorithmic solution (Section 5). Our proofs are in general based on a consequence of classical A. Turing's result on undecidability of the well-known halting problem (Turing, 1936) which can be stated as follows.

Theorem *(Turing (1936)). Let M be a set with $1<|M|<\infty$. Then there is no algorithm which, for a given computable $c(0),c(1),\ldots$ taking values in M and any $a\in M$, determines whether the sequence contains an element equal to a.*

This article is a continuation of the thematic line of publications (Abramov et al., 2021; Abramov and Pogudin, 2023a; Ovchinnikov et al., 2020; Petkovšek, 2006; Pogudin et al., 2020; Wibmer, 2021) on difference equations with general infinite sequences as coefficients and/or solutions. Among these publications, the article by Petkovšek (2006) belongs to those works that provided a general basis for computer-algebraic studies of infinite sequences.

An abridged preliminary version of this paper appeared as a proceedings paper (Abramov and Pogudin, 2023b).

2. Possible value of dimension

Before passing to the main topic of the paper, algorithmic questions related to computing the dimension of the solution space of (1), we will show that in the case when the coefficients are computable sequences, there is no a priori relation between the order of the equations and the dimension of the solution space (in contrast to, say, the constant coefficient case).

Proposition 1. *For every $r\in\mathbb{Z}_{\geqslant 0}$ and $d\in(\mathbb{Z}_{\geqslant 0}\cup\{\infty\})$, there exists an equation of the form (1) of order r with both $a_0(n)$ and $a_r(n)$ not identical to zero which has d-dimensional solution space.*

Before proceeding to the proof of the proposition, we introduce one useful construction: for difference equations (1), we define *interlacing* as follows. Consider two such equations and assume that they both have order r, since this can be achieved by adding several zero coefficients. Denote their coefficients by $a_0(n),\ldots,a_r(n)$ and $b_0(n),\ldots,b_r(n)$, respectively. We define sequences $c_0(n),\ldots,c_r(n)$ as follows

$$c_i(n):=\begin{cases}a_i(n/2), & \text{if } n \text{ is even,}\\ b_i((n-1)/2), & \text{if } n \text{ is odd.}\end{cases}$$

Now we define the interlacing of the original equations (we will denote it by the direct sum sign $\oplus$) as the following equation of order $2r$:

$$c_r(n)y(n+2r)+c_{r-1}(n)y(n+2r-2)+\ldots+c_1(n)y(n+2)+c_0(n)y(n)=0. \tag{2}$$

By construction, the solutions of the equations (2) are exactly sequences of $y(n)$ for which $y(2n)$ is a solution to the first equation and $y(2n+1)$ is a solution to the second.

In particular, the dimension of the solution space of the constructed equation is the sum of the dimensions of the solution spaces of the original equations. Interlacing of more than two terms is defined analogously.

Proof of Proposition 1. For $d \in (\mathbb{Z}_{\geqslant 0} \cup \{\infty\})$, we define a sequence $w_d(n)$ such that $w_d(n) = 0$ for $0 \leqslant n < d$ and $w_d(n) = 1$ otherwise. We denote by E_d the equation $w_d(n)y(n) = 0$. We note that the dimension of the solution space of E_d is equal to d. Equations E_d prove the lemma for the case $r = 0$, so we will further focus on the case $r > 0$.

For $r > 0$, we define an equation E_r° by $(1 - w_1(n))y(n) + y(n + r) = 0$. For every $n \neq 0$, it implies that $y(n + r) = 0$, so $y(n) = 0$ for every $n \neq r$. Furthermore, by taking $n = 0$, we obtain $y(0) + y(r)$ which, together with $y(0) = 0$ established earlier, implies $y(r) = 0$. Thus, the only solution of E_r° is the zero solution. Now we consider an equation $E_d \oplus E_r^\circ$. It has order r with the leading and trailing coefficients being not identically zero, and it has d-dimensional solution space. □

3. Existence of nonzero solutions

Proposition 2. *Let*

$$v(0), v(1), \ldots \tag{3}$$

be a computable sequence. Then a first-order difference equation (which will be called signal*) can be presented, the dimension of the solution space of which is equal to* 1 *if* (3) *is identically zero, and is equal to* 0 *(i.e., the equation has no nonzero solutions) otherwise.*

Proof. Based on the computable sequence $v(n)$, we define a computable two-sided sequence $w(n)$:

$$w(n) := \begin{cases} 1, \text{ if } n < 0, \\ 1, \text{ if } n \geqslant 0 \text{ and } v(k) = 0 \text{ for all } k = 0, 1, \ldots, n, \\ 0, \text{ if } n \geqslant 0 \text{ and } v(k) \neq 1 \text{ for some } k \text{ such that } 0 \leqslant k \leqslant n. \end{cases}$$

By construction the sequence $w(n)$ consists of only ones if and only if the sequence $v(n)$ consists of only zeros. If the sequence $v(n)$ contains at least nonzero element, then there is n_0 such that $w(n) = 1$ for $n \leqslant n_0$ and $w(n) = 0$ for $n > n_0$. In the first case, the equation $y(n + 1) - w(-n)y(n) = 0$ has a solution space of dimension 1 (all constant sequences and only them will be the solutions), in the second case, the solution space has dimension 0 (the equation has no nonzero solutions). □

The following theorem is a direct consequence of Proposition 2.

Theorem 1. *(i) There is no algorithm that tests the existence of a nonzero solution to a given equation.*

(ii) There is no algorithm that computes the dimension of the solutions space of a given equation.

Proof. An algorithm that tests for the presence of nonzero solutions to a given equation would make it possible to check for the presence of nonzero elements in a given computable sequence $v(0), v(1), \ldots$, which contradicts Turing's result (Turing, 1936). □

4. Computing dimension with a priori knowledge

In this section, we prove that even with some a priori restrictions on the dimension of the solution space, the problem of determining the exact dimension is still undecidable.

Theorem 2. *For any subset $\mathcal{S} \subseteq (\mathbb{Z}_{\geqslant 0} \cup \{\infty\})$ with $|\mathcal{S}| > 1$, there is no algorithm that computes the dimension of the solution space d for a given equation of the form (1), for which it is known in advance that $d \in \mathcal{S}$.*

Proof. Consider two distinct elements a and b from $\mathcal{S}$. For an arbitrary computable sequence $v(0)$, $v(1), \dots$ we will construct an equation with the dimension of the solution space being b if $v(n)$ is identically zero and being a otherwise. Then the undecidability of the problem of determining the dimension from the set $\{a, b\} \subseteq \mathcal{S}$ will follow from Turing's result (Turing, 1936).

Consider the case $a, b \neq \infty$, and let $b > a$. Then we consider the equation E_0 having an a-dimensional space of solutions (for example, any equation of order a with constant nonzero coefficients) and the equation E_1 constructed from $v(n)$ in Proposition 2. Consider the equation $E_0 \oplus \underbrace{E_1 \oplus \ldots \oplus E_1}_{b-a \text{ times}}$ (recall that $\oplus$ stands for interlacing). The dimension of its solution space is equal to b if $v(n)$ is identically zero and a otherwise. Which is what we aimed at.

Consider now the case when one of a and b is equal to infinity, let it be a. We construct a sequence $w(n)$ such that $w(n) = 1$ for $n < 0$, $w(n) = 1$ for $n \geqslant 0$ if all $v(0), \dots, v(n)$ are zeros and $w(n) = 0$ otherwise. We define the equation E_2 as $w(n)y(n) = 0$. If all elements of $v(n)$ are zeros, then $w(n) \equiv 1$, and hence the only solution is zero. If a nonzero element occurs in $v(n)$, then there are infinitely many zeros in $w(n)$, and hence the dimension of the space of solutions will be infinite-dimensional. Let the equation E_3 be any equation that has an b-dimensional space of solutions. Then $E_2 \oplus E_3$ will be the desired equation. □

Corollary 1. *For any non-negative integer k, there is no algorithm that would check if the solution space of the equation* (1) *has dimension k.*

Proof. If such an algorithm existed, then it could be used to calculate the dimension of the solution space in the case when it is known that the solution space is contained in the set $\mathcal{S} = \{k,\ k+1\}$. This would contradict Theorem 2. □

5. A case of decidability

Consider the case when the sequences $a_0(n), \dots, a_r(n)$ are in fact periodic. We will show that in this case, the dimension of the solution space can be computed using some standard tools from computer algebra, and the rest of the section will be devoted to proving the following proposition.

Proposition 3. *There is an algorithm which takes as input periodic sequences $a_0(n), \dots, a_r(n)$ and computes the dimension of the solution space of the equation*

$$a_r(n)y(n+r) + \ldots + a_1(n)y(n+1) + a_0(n)y(n) = 0$$

in the ring of two-sided sequences.

Consider a positive integer $H > r$ such that the lengths of the periods of $a_0(n), \dots, a_r(n)$ divide H (such H can always be taken to be a large enough common multiple of the period lengths). We will "decompose" $y(n)$ into H sequences $y_0(n) := y(Hn)$, $y_1(n) := y(Hn+1)$, $\dots$, $y_{H-1}(n) := y(Hn + H - 1)$. Then the original equation (1) translates into the following H linear difference equations with constant coefficients:

1. $a_0(i)y_i(n) + \ldots + a_r(i)y_{i+r}(n) = 0$ for $0 \leqslant i < H - r$;
2. $a_0(i)y_i(n) + \ldots + a_{H-1-i}(i)y_{H-1}(n) + a_{H-i}(i)y_0(n+1) + \ldots + a_r(i)y_{i+r-H}(n+1) = 0$ for $H - r \leqslant i < H$.

This way we have reduced the problem of computing the dimension of the solution space of (1) to the problem of computing the dimension of the solution space of the system above. We will state and solve this problem in a slightly more general form: for given $\ell \times H$ matrices A_0 and A_1, determine the dimension of the solution space of the system

$$A_0 \cdot (y_0(n),\ \dots,\ y_{H-1}(n))^T + A_1 \cdot (y_0(n+1),\ \dots,\ y_{H-1}(n+1))^T = 0. \tag{4}$$

In order to do this, we consider a free module F over the ring of Laurent polynomials $\mathbb{Q}[t, t^{-1}]$ with H generators $e_0, \ldots, e_{H-1}$. Let M be a submodule of F generated by the entries of

$$(A_0 + tA_1) \cdot (e_0, \ \ldots, \ e_{H-1})^T. \tag{5}$$

Lemma 1. *The dimension of the solution space of* (4) *is equal to the dimension of the quotient module F/M over $\mathbb{Q}$.*

Proof. Let $S := F/M$. We will define a linear bijective map between the solutions of (4) and linear functionals $S \to \mathbb{Q}$. For a functional $\varphi\colon S \to \mathbb{Q}$, we define $y_i(n) = \varphi(t^n e_i)$. Then the generators of M and their translations by integer powers of t will imply the equalities (4).

In the other direction, assume that we are given a solution of (4). We define $\widetilde{\varphi}\colon F \to \mathbb{Q}$ by $\widetilde{\varphi}(t^n e_i) = y_i(n)$. Since the sequences $y_0(n), \ldots, y_{H-1}(n)$ satisfy (4), we have that $\widetilde{\varphi}(M) = 0$. Thus, $\widetilde{\varphi}$ induces a well-defined linear functional on the quotient module F/M. □

Thanks to Lemma 1, the question of determining the dimension of the solution space of an equation (1) with periodic coefficients reduces to computing the dimension of the corresponding finitely presented module over the ring of Laurent polynomials. The latter problem can be solved using Gröbner basis or using the Hermite normal form over the Laurent polynomial ring.[1] In order to keep this note self-contained, we will present one way of computing this dimension based on a standard construction reducing the study of a module to the study of an ideal in a polynomial ring.

Lemma 2. *In the notation of this section, we consider a polynomial ring $\mathcal{R} = \mathbb{Q}[t_1, t_{-1}, x_0, \ldots, x_{H-1}]$ and an ideal*

$$I := \langle t_1 t_{-1} - 1, \ (A_0 + t_1 A_1) \cdot (x_0, \ldots, x_{H-1})^T, \ \{x_i x_j \mid 0 \leqslant i, j < H\}\rangle.$$

Then the $\mathbb{Q}$-dimension of the quotient module F/M is equal to the $\mathbb{Q}$-dimension of the ideal generated by the images of $x_0, \ldots, x_{H-1}$ in $\mathcal{R}/I$.

Proof. Let J be the ideal in $\mathcal{R}$ generated by I and $x_0, \ldots, x_{H-1}$. We will denote its image in $\mathcal{R}/I$ by $\widetilde{J}$ which can be also viewed as an $\mathcal{R}/I$-module. Furthermore, since the multiplication by $x_0, \ldots, x_{H-1}$ induces zero operator on $\widetilde{J}$, $\widetilde{J}$ is in fact a module over $\mathcal{R}/J \cong \mathbb{Q}[t_1, t_{-1}]/\langle t_1 t_{-1} - 1\rangle$. This ring is isomorphic to the ring of Laurent polynomials $\mathbb{Q}[t, t^{-1}]$ with the isomorphism given by $t_1 \to t$, $t_{-1} \to t^{-1}$. We define a homomorphism of modules over the Laurent polynomial ring $\varphi\colon F \to \widetilde{J}$ by $\varphi(e_i) = x_i$ for every $0 \leqslant i < H$. We observe that $\operatorname{Ker}(\varphi) \supset M$, so this homomorphism induces a surjective homomorphism $\widetilde{\varphi}\colon F/M \to \widetilde{J}$. Moreover, since I contains all the degree two monomials in x's, and linear relations on x's are exactly (5), $\widetilde{\varphi}$ is an isomorphism. The existence of such an isomorphism implies the equality of dimensions. □

Using Lemma 2, the dimension of F/M can be determined by computing a Gröbner basis of I and counting the monomials divisible by at least one of $x_0, \ldots, x_{H-1}$ but not divisible by any of the leading monomials of the basis. This completes the proof of Proposition 3.

Finally, we would like to point out that every periodic sequence satisfies a linear recurrence with constant coefficients (in other words, belongs to the class of C-finite sequences), so the class of sequences considered in this section is closely related with C^2-finite sequences studied in Jiménez-Pastor et al. (2023). More precisely, C^2-finite sequence is a solution of (1) with the coefficients being C-finite such that the leading coefficient $a_r(n)$ does not contain zeros. The latter condition considerably simplifies the problem of computing the dimension of the solution space (in particular, implies that this dimension is always finite). A natural generalization of the problem studied in this section

[1] We would like to thank Manuel Kauers for suggesting the approach via HNF.

would be a problem of computing the dimension of the solution space of (1) with all coefficients being C-finite (that is, without requiring the absence of zeros in $a_r(n)$). We do not know if this problem is algorithmically decidable.

CRediT authorship contribution statement

Sergei Abramov: Writing – review & editing, Writing – original draft, Investigation. **Gleb Pogudin:** Writing – review & editing, Writing – original draft, Investigation.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Acknowledgements

We would like to thank the referees for careful reading and detailed comments which helped us improve the manuscript.

References

Abramov, S., Pogudin, G., 2023a. Linear difference operators with sequence coefficients having infinite-dimensional solution spaces. ACM Commun. Comput. Algebra 57, 1–4. https://doi.org/10.1145/3610377.3610378.

Abramov, S., Pogudin, G., 2023b. On the solution space of linear difference equations over the ring of computable sequences. In: Proceedings of the XIV Conference "Differential Equations and Related Topics". Kolomna, Moscow Region, June 16–17, pp. 9–15. In Russian.

Abramov, S., Barkatou, M.A., Petkovšek, M., 2021. Linear difference operators with coefficients in the form of infinite sequences. Comput. Math. Math. Phys. 61, 1582–1589. https://doi.org/10.1134/s0965542521100018.

Jiménez-Pastor, A., Nuspl, P., Pillwein, V., 2023. An extension of holonomic sequences: C^2-finite sequences. J. Symb. Comput. 116, 400–424. https://doi.org/10.1016/j.jsc.2022.10.008.

Ovchinnikov, A., Pogudin, G., Scanlon, T., 2020. Effective difference elimination and Nullstellensatz. J. Eur. Math. Soc. 22, 2419–2452. https://doi.org/10.4171/jems/968.

Petkovšek, M., 2006. Symbolic computation with sequences. Program. Comput. Softw. 32, 65–70. https://doi.org/10.1134/s0361768806020022.

Pogudin, G., Scanlon, T., Wibmer, M., 2020. Solving difference equations in sequences: universality and undecidability. Forum Math. Sigma 8. https://doi.org/10.1017/fms.2020.14.

Turing, A.M., 1936. On computable numbers, with an application to the Entscheidungsproblem. Proc. Lond. Math. Soc. 2, 230–265.

Wibmer, M., 2021. On the dimension of systems of algebraic difference equations. Adv. Appl. Math. 123, 102136. https://doi.org/10.1016/j.aam.2020.102136.