The Conference "Computer Algebra" in Moscow

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The International Conference "Computer Algebra" was held in Moscow, Russia from June 17-21, 2019. The conference web-site is http://www.ccas.ru/ca/conference. Co-organized by the Dorodnicyn Computing Centre (Federal Research Center "Computer Science and Control") of Russian Academy of Sciences and the Peoples' Friendship University of Russia, it was devoted to computer algebra and related topics.

This is the third edition of the conference, and the previous two were in 2016 and 2017, respectively.

1   Invited Talk Abstracts

Experience in using the Fast Automatic Differentiation to solve inverse coefficient problems

A. P. Albu (Dorodnicyn Computing Center, FRC CSC of RAS, Russia; Moscow Institute of Physics and Technology, Russia), *Y. G. Evtushenko* (Dorodnicyn Computing Center, FRC CSC of RAS, Russia; Moscow Institute of Physics and Technology, Russia), *V. I. Zubov* (Dorodnicyn Computing Center, FRC CSC of RAS, Russia; Moscow Institute of Physics and Technology, Russia)

The Fast Automatic Differentiation methodology (FAD-methodology) is a modern approach to help you effectively solve optimal control problems of complex dynamic systems. Using this methodology, the authors had been resolved several interesting theoretically and practically important problems. This paper demonstrates the use of FAD-methodology to solve inverse coefficient problems. The importance of these problems due to the need to create new materials, some characteristics of which are not known in advance, and you can identify them by solving inverse problems. The inverse problem considered is devoted to the determination of temperature-dependent thermal conductivity of the substance according to the results of a pilot monitoring of temperature field in the investigated substance and heat flux on the surface of an object. The consideration is based on the initial boundary value problem for the non-stationary heat equation. The inverse coefficient problem is reduced to the variational problem: it is required to find such dependence of the thermal conductivity coefficient on the temperature, at which the temperature field and heat fluxes at the boundary of the object, obtained as a result of solving the direct problem, differ little from

the data obtained experimentally. The algorithm for the numerical solution of the posed inverse problem based on the modern FAD-methodology was proposed. The examples of solving the inverse coefficient problem confirm the efficiency of the proposed algorithm.

An efficient algorithm for the simultaneous triangularization of a finite set of matrices

M. Barkatou (University of Limoges, CNRS, France), *T. Cluzeau* (University of Limoges, CNRS, France)

In the study of linear differential systems, one can be interested in deciding whether a set of m given square matrices $A_1, \ldots, A_m$ are simultaneously triangularizable or not. If the answer is yes, then we sometimes need to compute effectively an invertible matrix P such that, for all $i \in \{1, \ldots, m\}$, the matrix $P^{-1} A_i P$ is upper triangular. See, for instance, the recent paper by Barkatou and Gontsov (Linear differential systems with small coefficients: various types of solvability and their verification. https://arxiv.org/abs/1901.09951, 2019).

The classical approach consists in using Lie algebra theory to test whether the matrix Lie algebra spanned by the A_i's is solvable (e.g., using the so-called derived series) and if so, find a basis in which all matrices of the Lie algebra are upper triangular using a constructive version of Lie's theorem on solvable algebras for computing common eigenvectors. See the book by De Graaf (Lie Algebras: Theory and Algorithms. Volume 56 of North-Holland Mathematical Library. Elsevier, 2000).

In this presentation, we will rather consider the following result due to McCoy (On the characteristic roots of matric polynomials. Bull. Amer. Math. Sot. 42: 592-600, 1936): matrices $A_1, \ldots, A_m$ are simultaneously triangularizable if and only if, for every scalar polynomial $p(x_1, \ldots, x_m)$ in the (non-commutative) variables $x_1, \ldots, x_m$, each of the matrices $p(A_1, \ldots, A_m)[A_i, A_j] = p(A_1, \ldots, A_m)(A_i A_j - A_j A_i)$ $(i, j = 1, \ldots, m)$ is nilpotent. We shall show that the proof of this result provided in (M.P.Drazin, J.W.Duugey, K.W.Gruenberg. Some theorems on commutative matrices, *J. London Math. Sot.* 26: 221-228, 1951) can be turned into an efficient algorithm for computing particular common eigenvectors of $A_1, \ldots, A_m$. As a consequence, this yields an efficient algorithm for the simultaneous triangularization problem. Note that this new approach does not require the construction of the Lie algebra spanned by the matrices A_i's. The algorithm has been implemented in Maple and we will show comparisons to the implementation of the "Lie algebra method" included in the `DifferentialGeometry/LieAlgebras` package of Maple.

Normal form of the periodically perturbed Hamiltonian system

A. D. Bruno (Keldysh Institute of Applied Mathematics of RAS, Russia)

Near a stationary solution we consider the Hamiltonian system with such perturbation, that the unperturbed Hamiltonian function is autonomous and the perturbation of the Hamiltonian function is periodic in time. First we remind the normal form of the autonomous Hamiltonian function. Second we describe the normal form of the periodic perturbation of the Hamiltonian function. It can always be reduced to the time independent Hamiltonian. It allows to compute the local families of periodic solutions of the initial system. We also discuss problems of the computer algebra arising in these computations.

An additive Ore–Sato theorem on compatible rational functions

S. Chen (Academy of Mathematics and Systems Science, CAS, China), *J. Guo* (Academy of Mathematics and Systems Science, CAS, China)

The Ore–Sato theorem describes a multiplicative structure of multivariate hypergeometric terms. This theorem was first proved by Ore in 1930 in the bivariate case and then extended to the multivariate case by Sato in the 1960s via homological method. In 2002, Abramov and Petkovsek gave an elementary proof of the Ore–Sato theorem and proved a slightly corrected version of a conjecture of Wilf and Zeilberger on multivariate hypergeometric terms. Other different proofs of this theorem were also presented by Payne in 1997 and Hou in 2001 in their PhD dissertations. The continuous analogue of the Ore–Sato theorem was first obtained by Christopher in 1997 for bivariate compatible rational functions and later extended by Żołądek to the multivariate case in 1998. More recently, this result has been extended further to the multivariate continuous-discrete case by Chen et al. in 2011. The mixed extension of the Ore–Sato theorem has been used to prove the Wilf–Zeilberger conjecture on mixed hypergeometric terms by Chen and Koutschan in 2018. We will present an additive version of the Ore–Sato theorem which explores all possible rational Wilf–Zeilberger pairs.

On evaluation of Bessel functions of the first kind via Prony-like methods

A. Cuyt (University of Antwerp, Belgium), *W.-s. Lee* (University of Antwerp, Belgium; University of Stirling, UK), *Y. Ji* (East China Normal University, China), *M. Wu* (East China Normal University, China)

In this work, we study two variants of Prony's method on Bessel functions of first kind of integer order and obtain approximations as sums of sinusoidal functions, including sums of cosine terms and sums of sine terms, respectively. Experiments show the good performance of Prony-like methods. Further, an improvement is proposed by substitution of interpolation nodes with Chebyshev points.

Making many more matrix multiplication methods

M. Heule (University of Texas at Austin, U.S.), *M. Kauers* (Johannes Kepler University Linz, Austria), *M. Seidl* (Johannes Kepler University Linz, Austria)

Strassen's algorithm for matrix multiplication is based on the observation that the product of two 2×2 matrices can be computed with only 7 multiplications. It is known that there is no way to do it with fewer multiplications, and that Strassen's method is essentially the only way to do it with 7 multiplications. For 3×3-matrices, the situation is less clear. More than 40 years ago, Laderman gave a method that uses 23 multiplications, and still nobody knows whether there is a way to do it with fewer multiplications. Laderman's method is not unique. Several authors have later presented isolated additional methods that work for arbitrary coefficient rings, and there is even a family of methods with three free parameters but restricted to coefficient rings containing $\mathbb{Q}$. Using SAT solvers and Gröbner bases technology, we have found more than 10000 pairwise inequivalent new matrix multiplication methods with 23 multiplications. We were able to cluster them together into families with up to 16 parameters which apply without any restriction on the coefficient domain. In conclusion, the set of methods with 23 multiplications appears

to be much larger than it seemed to be until now, and it is quite possible that the methods we found are still just the top of the iceberg. Although our results have no immediate implications on the complexity of matrix multiplication, we do hope that the vast amount of new methods will eventually contribute to a better understanding of why these methods exist at all, and whether 23 is best possible for 3×3.

Interpolating reduced data based on piecewise-cubic splines

R. Kozera (Warsaw University of Life Sciences, Poland; John Paul II Catholic University of Lublin, Poland), *M. Wilkołazk* (Warsaw University of Life Sciences, Poland)

The problem of fitting reduced data Q_m by C^k piecewise-cubics $\hat{\gamma}$ is discussed for $k = 0, 1$. Here the set Q_m forms a sequence of m interpolation points $q_i = \gamma(t_i)$ in arbitrary Euclidean space $\mathbb{E}^n$. In our setting the data points Q_m are not accompanied by the corresponding interpolation knots obeying $t_0 = 0 < t_1 < ... < t_m = T$. We review the existing results determining the asymptotics in estimating the unknown curve γ by $\hat{\gamma}$ with the aid of the so-called exponential parameterization. The latter represents a family of knots' estimates $\hat{t}_i^\lambda \approx t_i$ (fulfilling $\hat{t}_0^\lambda = 0 < \hat{t}_1^\lambda < ... < \hat{t}_m^\lambda = \hat{T}$) expressed in terms of a single parameter $\lambda \in [0, 1]$ and Q_m. In order to enforce the convergence in $\gamma \approx \hat{\gamma}$ the mapping $\phi : [0, T] \to [0, \hat{T}]$ needs to be appropriately chosen. With the aid of computer algebra and analytic arguments this paper addresses the following issues: *a)* the *sharpness* of the convergence rates in estimation $\gamma \approx \hat{\gamma}$ is demonstrated, *b)* the *necessity of more-or-less uniformity* of Q_m and of the *regularity* of γ in proving the latter is justified to be essential, *c)* the *sufficient conditions* for ϕ to form *a genuine parameterization* are formulated in terms of algebraic inequalities which can be visualized with the aid of Mathematica.

Hilbert series of noncommutative algebras and context-free languages

R. La Scala (Università di Bari, Italy), *D. Piontkovski* (National Research University Higher School of Economics, Russia), *S. K. Tiwari* (Defence Research & Development Organization, India)

The Hilbert series is one of the most important algebraic invariants of infinite-dimensional graded associative algebra. The noncommutative Groebner basis machine reduces the problem of finding Hilbert series to the case of monomial algebra.

We apply both noncommutative and commutative Groebner bases theory as well as the theory of formal languages to provide a new method for symbolic computation of Hilbert series of graded associative algebras. Whereas in general the problem of computation oh such a Hilbert series is known to be algorithmically unsolvable, we have describe a general class of algebras (called *homologically unambiguous*) with unambiguous context-free set of relations for which our method give effective algorithms. Unlike previously known methods, our algorithm is applicable to algebras with irrational Hilbert series and produces an algebraic equation which defines the series. The examples include infinitely presented monomials algebras as well as finitely presented algebras with irrational Hilbert series such that the associated monomial algebras are homologically unambiguous.

Inverse Zeilberger's problem

M. Petkovšek (University of Ljubljana, Slovenia)

Given a sum of the form $s_n = \sum_{k=0}^n F(n,k)$ where $F(n,k)$ is proper hypergeometric, Zeilberger's algorithm returns a linear recurrence with rational coefficients satisfied by s_n. Here we consider the inverse problem: Given a linear recurrence operator L with polynomial coefficients, find solutions that have the form of a definite sum. As a small first step towards its solution, we present an algorithm which, given L and a product of binomial coefficients of the form $F(n,k) = \prod_{i=1}^m \binom{a_i n + b_i}{k}$ with $a_i \in \mathbb{N} \setminus \{0\}$, returns a linear recurrence operator L' with rational coefficients such that $L(\sum_{k=0}^\infty F(n,k) h_k) = 0$ if and only if $L'h = 0$.

Geometric integrators in mechanics – the need for computer algebra tools

V. Salnikov (CNRS & La Rochelle University, France), *A.Hamdouni* (La Rochelle University, France)

In this contribution we will describe some objects of the generalized geometry that appear naturally in the qualitative analysis of mechanical systems. In particular we will discuss the Dirac structures within the framework of the systems with constraints as well as of the port-Hamiltonian systems.

From the mathematical point of view, Dirac structures generalize simultaneously symplectic and Poisson structures. As for mechanics, the idea is to design numerical methods that preserve these structures and thus guarantee good physical behaviour in the simulation.

Then, we will present a framework which is even more general – the one of differential graded manifolds (also called Q-manifolds), and discuss some possible ways of using them for the "structure preserving integrators" in mechanics.

For all of the mentioned constructions we will explain the problems that arise in generic situations – most of them are open, but we think they are suitable for handling with various computer algebra approaches.

The difference ring approach for symbolic summation

C. Schneider (Johannes Kepler University Linz, Austria)

A major breakthrough in symbolic summation was Michael Karr's algorithm (1981) that solves the following problem. Given a summand $f(k)$ in a $\Pi\Sigma$-difference field $\mathbb{F}$, decide constructively if there exists a $g(k)$ in $\mathbb{F}$ such that $f(k) = g(k+1) - g(k)$ holds. Given such a solution $g(k)$ and summing the telescoping equation over a valid summation range yields $\sum_{k=a}^b f(k) = g(b+1) - g(a)$. From the point of view of applications one is usually given only the summand $f(k)$ in terms of indefinite nested sums and products, and not a difference field $\mathbb{F}$ in which $f(k)$ can be represented properly. In this talk I will elaborate how indefinite nested sums defined over hypergeometric products can be always represented in $R\Pi\Sigma$-difference rings, an enhanced definition of Karr's $\Pi\Sigma$-fields. As a bonus we will obtain an algebraic machinery that can rewrite an expression in terms of indefinite nested sums and products to an equivalent expression such that among the arising sums and products (excepts elements such as α^k where α is a root of unity) no algebraic relations exist. Exploiting in addition the available difference ring/field algorithms for Zeilberger's creative telescoping and recurrence solving (utilizing as special cases Abramov's recurrence solver

for rational functions and Petkovsek's Hyper algorithm), I will demonstrate how non-trivial summation problems in combinatorics and particle physics can be tackled with the help of the summation package Sigma.

Computational aspects of irreducible polynomials

D. Ştefănescu (University of Bucharest, Romania)

We discuss some results on testing the irreducibility of polynomials and give some methods for constructing irreducible polynomials. We first survey classical methods of factorization of polynomials over the integers and irreducibility criteria that are based on properties of Newton's polygon.

Finally we give irreducibility criteria for univariate polynomials $F(X) = \sum_{i=0}^{d} a_i X^{d-i}$ over a discrete valuation domain in terms of Newton's polygon. We give applications to bivariate polynomials.

2 Contributed Talk Abstracts

An operator matrix having a given space of solutions

S. A. Abramov (Dorodnicyn Computing Center, FRC CSC of RAS, Russia), *M. Barkatou* (Université de Limoges, CNRS, France)

We consider differential full rank operator matrices over a differential field K of characteristic zero. The constant field of K is assumed to be algebraically closed. Together with each operator A we consider its solution space V_A; the components of solutions are supposed to belong to the universal Picard-Vessiot extension Λ of K. We prove that for any given finite set $F \subset \Lambda^m$ there exists a matrix whose solution space is generated by F (the entries of that matrix are, in general, in Λ, not necessarily in K).

Laurent solutions of linear ordinary differential equations with coefficients in the form of truncated power series

S. A. Abramov (Dorodnicyn Computing Center, FRC CSC of RAS, Russia), *D. E. Khmelnov* (Dorodnicyn Computing Center, FRC CSC of RAS, Russia), *A. A. Ryabenko* (Dorodnicyn Computing Center, FRC CSC of RAS, Russia)

Linear ordinary differential equations with formal power series coefficients represented in a truncated form are considered. We discuss the information on solutions belonging to the field of Laurent formal series which can be obtained from this representation of a given equation. We emphasize that we are interested in such information about solutions which is invariant with respect to possible prolongations of the truncated series representing the coefficients of the equation.

Linearizing differential equations: Riccati solutions as D^n-finite functions

M. Barkatou (Université de Limoges, CNRS, France), *A. Jiménez-Pastor* (Johannes Kepler University Linz, Austria)

D-finite (or holonomic) functions are a class of formal power series that satisfy linear differential equation with polynomial coefficients. The finite representation of these functions (using the differential equation and some initial conditions) boosted the development of algorithms working symbolically over them. This has been recently extended to the DD-finite class (functions satisfying linear differential equations with D-finite coefficients) and implemented some closure properties. It was also proved that DD-finite functions (and also their generalization to the D^n-finite functions) are differentially algebraic. In this document we show how solutions to non-linear differential equations (starting with the Riccati differential equation) are always D^n-finite functions for some n and proposed some ideas to set the difference between D^n-finite functions and differentially algebraic functions.

Bifurcations of periodic solutions to Hamiltonian system with discrete symmetries

A. B. Batkhin (Keldysh Institute of Applied Mathematics of RAS, Russia; Moscow Institute of Physics and Technology, Russia)

We consider an autonomous Hamiltonian system with two degrees of freedom, which system of canonical equations is t-invariant under Klein four-group of linear canonical automorphisms of the extended phase space of the system. Three types of bifurcations of a family of doubly symmetric periodic solutions: saddle-node bifurcation, pitch-fork bifurcation and period multiplying bifurcation, are investigated by means of simplified monodromy matrix of periodic solution. For last two types of bifurcations different scenarios are studied. All mentioned above kinds of bifurcations were found and investigated for the case of doubly symmetric periodic solutions of the Hill problem. Most computations were provided with CAS Maple.

Sparse interpolation, exponential analysis, Padé approximation, tensor decomposition

A. Cuyt (University of Antwerp, Belgium), *W.-s. Lee* (University of Antwerp, Belgium; University of Stirling, UK)

What is the connection of sparse interpolation from computer algebra with exponential analysis from signal processing, Padé approximation in approximation theory, and tensor decomposition in multilinear algebra? We explain that these seemingly unrelated topics are actually deeply intertwined.

While several of these connections deserve to be investigated more thoroughly, new results have been achieved based on the recently established relationships, and explored in different applications.

Symbolic-numeric implementation of the four potential method for calculating normal modes: an example of square electromagnetic waveguide with rectangular insert

D. V. Divakov (Peoples' Friendship University of Russia, Russia), *A. A. Tiutiunnik* (Peoples' Friendship University of Russia, Russia), *M. D. Malykh* (Peoples' Friendship University of Russia, Russia), *L. A. Sevastianov* (Joint Institute for Nuclear Research, Russia)

In this paper, the Maple computer algebra system is used to construct a symbolic-numeric implementation of the method for calculating normal modes of square closed waveguides in a vector formulation. The method earlier proposed by Malykh et al. [M.D.Malykh, L.A.Sevastianov, A.A.Tiutiunnik, N.E.Nikolaev. On the representation of electromagnetic fields in closed waveguides using four scalar potentials, *Journal of Electromagnetic Waves and Applications*, 32 (7), 886-898 (2018)] will be referred to as the method of four potentials. The Maple system is used at all stages of treating the system of differential equations for four potentials: the generation of the Galerkin basis, the substitution of approximate solution into the system under study, the formulation of a computational problem, and its approximate solution.

Thanks to the symbolic-numeric implementation of the method, it is possible to carry out calculations for a large number of basis functions of the Galerkin decomposition with reasonable computation time and then to investigate the convergence of the method and verify it, which is done in the present paper, too.

Algebraically mimetic finite difference approximations to PDE on regular orthogonal grids

V. P. Gerdt (Joint Institute for Nuclear Research, Russia)

In the given talk we discuss the algorithmic aspects of differential and difference algebra related to construction, on uniform and orthogonal grids, of finite difference approximations to polynomially-nonlinear partial differential equations and to systems of such equations. As this takes place, we impose on a finite difference approximation the condition of *strong consistency* with the differential equations. If the condition holds, then we call the approximation *algebraically mimetic*. This condition strengthens the commonly accepted requirement of *consistency* of differential ideal associated with the input differential the approximation, which we call *weak consistency*, and means approximability of any consequence of the input differential equation(s) by an element in the perfect difference ideal generated by the polynomials occurring in the finite difference approximation. In the case of linear equations one can use difference Gröbner bases for the elimination and differential Gröbner bases for the verification of strong consistency. If the input differential equations are nonlinear, then instead of differential Gröbner bases, which can be infinite, one can use the differential Thomas decomposition. Nonlinear difference Gröbner bases can also be infinite, and there are needs in design of mathematical methods, algorithms and software for construction of a difference analogue of the differential Thomas decomposition.

CAS pseudo-random numbers generators statistical test suits

M. N. Gevorkyan (Peoples' Friendship University of Russia, Russia), *D. S. Kulyabov* (Peoples' Friendship University of Russia, Russia; Joint Institute for Nuclear Research, Russia), *A. V. Demidova* (Peoples' Friendship University of Russia, Russia), *A. V. Korolkova* (Peoples' Friendship University of Russia, Russia)

For a long time the implementation of pseudo-random number sequence generators in standard programming language libraries and mathematical packages was of poor quality. The situation has started to improve relatively recently. Even nowadays a large number of libraries and poorly supported mathematical packages utilize the old algorithms of pseudo-random numbers generation. We describe four actual sets of statistical tests that can be used to test the generator that is used in a particular software system. The emphasis is on the use of command-line utilities, to avoid low-level C or C++ programming.

Algorithm for calculating interpolation Hermite polynomials in d-dimensional hypercube in the analytical form

A. A. Gusev (Joint Institute for Nuclear Research, Russia), *G. Chuluunbaatar* (Joint Institute for Nuclear Research, Russia; Peoples' Friendship University of Russia, Russia), *O. Chuluunbaatar* (Joint Institute for Nuclear Research, Russia; National University of Mongolia, Mongolia), *V. P. Gerdt* (Joint Institute for Nuclear Research, Russia), *S. I. Vinitsky* (Joint Institute for Nuclear Research, Russia; Peoples' Friendship University of Russia, Russia), *L. L. Hai* (Ho Chi Minh City University of Education, Vietnam), *T. T. Lua* (Ho Chi Minh City University of Education, Vietnam), *V. L. Derbov* (Saratov State University, Russia), *A. Góźdź* (University of M. Curie-Skłodowska, Poland)

A new symbolic algorithm implemented in Maple for constructing Hermitian finite elements in the standard d-dimensional hypercube is presented. The basis functions of finite elements are polynomials, determined from a specially constructed set of values of the polynomials themselves and their first partial derivatives at the corners of the hypercube. Such a choice of values allows one to construct a piecewise polynomial basis continuous on the boundaries of finite elements together with the derivatives up to the first order. In the case of a d-dimensional cube, it is shown that the basis functions are determined by a product of d interpolation Hermite polynomials depending on each of the d variables with continuous first derivatives on the boundaries of finite elements. It can be used to solve elliptic boundary value problems by means of the finite element method. The efficiency and accuracy order of the finite element scheme, algorithm and program are demonstrated by the example of an exactly solvable Helmholtz problem for a three-dimensional cube.

Application of computer algebra methods for investigation of the stationary motions of the system of two connected bodies moving along a circular orbit

S. A. Gutnik (Moscow State Institute of International Relations, Russia; Moscow Institute of Physics and Technology, Russia); *V. A. Sarychev* (Keldysh Institute of Applied Mathematics RAS, Russia)

Computer algebra and numeric methods are used to investigate properties of a nonlinear algebraic system that determines the equilibrium orientations for a system of two bodies connected by a spherical hinge that moves along a circular orbit under the action of gravitational torque. The main attention is paid to study the conditions of existence of the equilibrium orientations for the system of two bodies for special cases, when one of the principal axes of inertia both the first and second body coincides with the normal to the orbital plane, with radius vector or tangent to the orbit. To determine the equilibrium orientations for the system of two bodies, the system of 12 stationary algebraic equations is decomposed into 9 subsystems. The computer algebra method based on the algorithm for the construction of a Gröbner basis applied to solve the stationary motion system of algebraic equations. Depending on the parameters of the problem,

the number of equilibria is found by numerical analysis of the real roots of the algebraic equations from the Gröbner basis constructed.

Computing primitive idempotents in centralizer ring of permutation representation of wreath product

V. V. Kornyak (Joint Institute for Nuclear Research, Russia)

An approach to compute the complete set of primitive orthogonal idempotents in the centralizer ring of the permutation representation of a wreath product is described. This set determines the decomposition of the representation into irreducible components. In quantum mechanics, these idempotents manifest themselves as projection operators into irreducible invariant subspaces of the Hilbert space of a multipartite quantum system. The C implementation of the approach is able to construct irreducible decompositions of high-dimensional and high-rank representations of wreath products.

Canonical representation of monomials with indices and contractions

A. P. Kryukov (Moscow State University, Russia), *G. B. Shpiz* (Moscow State University, Russia)

The article deals with the problem of reducing monomials with indices to canonical form. This problem is met in many fields of mathematics and physics and is one of the key algorithmic problems of computer algebra. The most commonly used approach to the problem is based on a double coset. However, this approach has some limitations. For example, it cannot be used for expressions in which have linear relationships with more then two terms. The article provides a definition of the canonical form for monomials from indexed factors, taking into account summation indices and linear relations for them. Linear relations can include two and more terms with different order of indices. Also a special colored graph associated with monomials is introduced and formulate the problem of calculating the canonical form of a monomial in terms of the canonical numbering of such graphs. The final part of the paper presents an algorithm for reducing monomials to canonical form, based on the calculation of the set of canonical numbering of edges of a color graph conjugated to the monomial. The algorithm does not use the construction of adjacency classes for the group renaming summation indices. The computational complexity of the proposed algorithm does not depend on the number of summation indices, but on the number of automorphisms of the color graph and is small for expressions that occur in practice.

Linearizability of differential equations: theory and algorithms

D. Lyakhov (King Abdullah University of Science and Technology, Saudi Arabia)

Solving nonlinear differential equations is one of the fundamental and practically important research challenges in mathematics. However, the problem of their algorithmic linearizability so far remained unsolved. In this contribution, we propose a solution of this problem for a wide class of nonlinear ordinary differential equation of arbitrary order. Criteria for linearizability of scalar ordinary differential equations is based on algorithmic calculus of determining Lie symmetry equations. Two main points are analysis of dimension

of determining system and construction of derived algebra for abstract Lie symmetry algebra. New recent generalization for PDE will be also discussed.

On a provable program for generic arithmetic of fractions

S. D. Meshveliani (Ailamazyan Program Systems Institute of RAS, Russia)

There are described the design principles for certified programs for arithmetic of fractions over any domain with the greatest common divisor function. This is a small part of the library `DoCon-A` of certified programs for a computer algebra library designed by the author. In this system, programs include definitions for the corresponding mathematical notions and proofs for the main properties of the implemented methods. These proofs are checked by the compiler. It is used a purely functional programming language `Agda`, which also supports the feature of *dependent types*. It is described the technique for providing formal machine-checked proofs for a certain optimized method to sum fractions.

An algorithm for solving a family of diophantine equations of degree four which satisfy Runge's condition

N. N. Osipov (Siberian Federal University, Russia), *A. A. Kytmanov* (Siberian Federal University, Russia)

In modern computer algebra systems (such as Mathematica, Maple, SageMath, etc.) there is a small number of implemented algorithms for solving diophantine equations in integers. In this paper we suggest an algorithmic implementation of elementary version of Runge's method for a family of diophantine equations of degree four. Although Runge's method has been known for more than 100 years, its implementation in computer algebra systems (CAS) is very limited (due to very large theoretical upper bounds for solutions which is useless for practical computer implementation via brute force method).

The elementary version of Runge's method for diophantine equations of small degree is based on a convenient parametrization which provides enumerating possible integer solutions. In some cases this leads to the practically working solving algorithms for the diophantine equations of degree three and four.

The software implementation of the proposed solving method (in its optimized version) is planned in PARI/GP CAS.

Close-packed dimers on a cylinder boundary. Derivation of generating functions in the Maple system

S. N. Perepechko (Petrozavodsk State University, Russia)

The problem of counting the number of perfect matchings on the cylinders $C_m \times P_n$ is considered for the case when the matching contains a given edge on the boundary of the cylinder. For fixed values of $3 \leqslant m \leqslant 14$ a set of recurrence relations and associated generating functions is constructed. Application of the obtained results to the dimer problem allowed us to deduce closed form expressions for the probabilities of occupation a given edge of the lattice with a dimer when $n \to \infty$. A set of corrections to the limit values of probabilities due to the finite order of the graph is calculated.

Symbolic computation in studying the dynamics of generalized Atwood's machine

A. N. Prokopenya (Warsaw University of Life Sciences, Poland)

A generalized model of the Atwood machine when a pulley of finite radius is replaced by two small separate pulleys and one body can swing in a vertical plane is considered. Doing necessary symbolic calculations, we have obtained equations of motion of the system and analyzed their solutions in case of small oscillations. We have shown that in case of small difference of masses of the bodies there exists a state of dynamical equilibrium of the system when the oscillating body acts like a pendulum whose length undergoes small oscillation. The corresponding periodic solutions of the equations of motion are obtained in the form of power series in terms of a small parameter. Comparison of the obtained results with the corresponding numerical solution of the equations of motion demonstrates its validity. All the relevant calculations are performed with the computer algebra system Wolfram Mathematica.

Effective Schrödinger equation for one-dimensional potential box with rapidly oscillating width

N. P. Tretyakov (Russian State Social University, Russia; Russian Presidential Academy of National Economy and Public Administration, Russia)

The method of the effective Schrödinger equation is applied to the one-dimensional motion of a particle in a potential box with a potential function and high-frequency time-dependent boundary conditions. It was assumed that the oscillation amplitudes are small in comparison with the potential box width. The resulting system can be considered as a particle in a static potential box, located in the field of some effective potential, different from the initial one. In this work, we consider a particular case where the width of the box oscillates, while its center does not move, with the subsequent construction of the effective Schrödinger equation. Oscillations of width lead to additional terms in the effective equation proportional not only to the second derivative of the potential, but to the first derivative as well. This means that effects from fast external driving can be observed only in cases where the potentials have regions with large values of $V''(x)$ and $V'(x)$. Another fundamental difference is that in additional terms the first and second derivatives are multiplied by x and x^2. In the case of width oscillations, compared to the oscillations of the box as a whole (this case was previously calculated elsewhere), the changes are even more significant and striking. In particular, the potential barrier may turn into a potential well (and vice versa).

Non-local singularities on families of periodic solutions to ODEs

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We consider degenerate solutions on families of periodic solutions to ODEs. The degeneracy can mean any property of the solution that isolates the solution from the generic case. This can mean a bifurcation or a topological peculiarity on the family that causes a failure of a numerical algorithm that was used for generic cases. We suggest a method of computation of such degeneracies with variational equations of higher order with the same accuracy as ordinary solutions.

MAPLE program for calculating transition probabilities of hydrogen-like atoms in quantum mechanics with non-negative distribution function

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The program is proposed for a realization of the symbolic algorithm based on the quantum mechanics with non-negative probability distribution function (QDF) and for calculations of transition probabilities for hydrogen-like atoms. Transition probabilities are calculated and compared with experimental data. Calculations of radial functions were used in calculations of oscillators strengths and transitional probabilities. These transition probabilities are calculated by the Galerkin method with the Sturm functions of the hydrogen atom as coordinate functions. Matrix elements of physical operators are required when the accurate theoretical determination of atomic energy levels, orbitals and radiative transition data need to be obtained for open-shell atoms and ions. It turns out that QDF seems to be equivalent to the traditional quantum mechanics in regard to predictions of experimental values. However, the existence of a phase-space probabilistic quantum theory may be an important advance towards the explanation and interpretation of quantum mechanics. Computer algebra methods seem to be absolutely necessary and indispensable for such calculations.