Dimensions of Solution Spaces of H-Systems*

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Abstract

An *H-system* is a system of first-order linear homogeneous recurrence equations for a single unknown function T, with coefficients which are polynomials with complex coefficients. We consider solutions of H-systems which are of the form $T: \operatorname{dom}(T) \to \mathbb{C}$ where either $\operatorname{dom}(T) = \mathbb{Z}^d$, or $\operatorname{dom}(T) = \mathbb{Z}^d \setminus S$ and S is the set of integer singularities of the system. It is shown that any natural number is the dimension of the solution space of some consistent H-system, and that in the case $d \geq 2$ there are H-systems whose solution space is infinite-dimensional. The relationship between dimensions of solution spaces in the two cases $\operatorname{dom}(T) = \mathbb{Z}^d$ and $\operatorname{dom}(T) = \mathbb{Z}^d \setminus S$ is investigated. We prove that every consistent H-system $\mathcal{H}$ has a non-zero solution T with $\operatorname{dom}(T) = \mathbb{Z}^d$. Finally we give an appropriate corollary to the Ore-Sato theorem on possible forms of solutions of H-systems in this setting.

1 Introduction

Systems of first-order linear homogeneous multivariate recurrence equations with polynomial coefficients and a single unknown function play a significant role in combinatorics as well as in the theory of hypergeometric functions. In

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applications, it is often not enough to consider solutions of such systems as abstract algebraic objects. Rather, the user is looking for a solution which is defined everywhere, or at least at all non-singular points of the system. If there are no singular points, the solution space is at most one-dimensional, and the solutions are easy to compute. However, in the presence of singularities the situation changes radically. In particular, the question of existence of a non-zero solution defined everywhere can be very non-trivial. In this paper, we investigate the possible values that the dimension of various solution spaces of such systems can have, a question important both from theoretical and algorithmic points of view.

More precisely, let $n_1, n_2, \ldots, n_d$ be variables ranging over the integers. We consider d-dimensional H-systems, that is to say,, systems of equations of the form

$$\begin{aligned} & f_i(n_1, n_2, \ldots, n_d)\, T(n_1, n_2, \ldots, n_i + 1, \ldots, n_d) \\ = \quad & g_i(n_1, n_2, \ldots, n_d)\, T(n_1, n_2, \ldots, n_d), \end{aligned}$$

where $f_i, g_i \in \mathbb{C}[x_1, x_2, \ldots, x_d] \setminus \{0\}$ are relatively prime polynomials for $i = 1, 2, \ldots, d$. The notion of singular points (singularities) of such systems can be defined in the usual way. Such singularities make obstacles (sometimes insuperable) for continuation of partial solutions of the system on all of $\mathbb{Z}^d$.

In this paper we consider two spaces of solutions of H-systems: the space V_1 of solutions defined everywhere on $\mathbb{Z}^d$, and the space V_2 of solutions that are defined at all nonsingular points of $\mathbb{Z}^d$. The precise definitions are given in Section 2 where it is also shown that the dimension of V_2 equals the number of components induced on $\mathbb{Z}^d$ by the singularities of the system.

In Sections 3, 4 and 5 we investigate the dimensions of the spaces V_1, V_2 and their relationship. It is well known [7] that if (in the case $d = 1$) one considers the germs of sequences at infinity (i.e., classes of functions $T : \mathbb{N}_0 \to \mathbb{C}$ which agree from some point on), the dimension of the solution space is 1. However, the situation is different with $\dim V_1$ and $\dim V_2$. In Section 3 we prove for the case $d = 1$ that if the equation has singularities then $1 \le \dim V_1 < \dim V_2 < \infty$, and that for any integers s, t such that $1 \le s < t$ there exists an equation with $\dim V_1 = s$ and $\dim V_2 = t$ (the case where there is no singularity is trivial: $\dim V_1 = \dim V_2 = 1$). In Section 4 we show that in the case $d > 1$ the possibilities are even richer: for any $s, t \in \mathbb{N} \cup \{\infty\}$ there exists an H-system with $\dim V_1 = s$ and $\dim V_2 = t$.

The central part of the paper is Section 5 where we prove that $\dim V_1 > 0$ for every consistent H-system. Thus we prove an existence theorem for H-systems, which claims that for every consistent H-system there is a non-zero solution defined everywhere on $\mathbb{Z}^d$. Since the coefficients $f_i(n_1, n_2, \ldots, n_d), g_i(n_1, n_2, \ldots, n_d)$, $i = 1, 2, \ldots, d$, can vanish for some $(n_1, n_2, \ldots, n_d) \in \mathbb{Z}^d$, this fact is not self-evident.

Example 1 Let $\mathcal{H}$ be the two-dimensional H-system

$$\begin{aligned} ((n_1+1)^2+n_2^2)T(n_1+1,n_2) &= (n_1^2+n_2^2)T(n_1,n_2), \\ (n_1^2+(n_2+1)^2)T(n_1,n_2+1) &= (n_1^2+n_2^2)T(n_1,n_2). \end{aligned}$$

The only singularity of $\mathcal{H}$ is the point $(0,0)$, and its space V_2 is spanned by

$$T(n_1,n_2) = \frac{1}{n_1^2+n_2^2}$$

which is defined and non-zero at all points of $\mathbb{Z}^2 \setminus \{(0,0)\}$. The situation is quite different with V_1. It is easy to check that any solution of $\mathcal{H}$ defined everywhere on $\mathbb{Z}^2$ vanishes at all points of $\mathbb{Z}^2$ except possibly at $(0,0)$, where its value can be chosen arbitrarily. So V_1 is spanned by

$$T(n_1,n_2) = \begin{cases} 1, & n_1 = n_2 = 0, \\ 0, & \text{otherwise.} \end{cases}$$

■

In the light of Example 1, it is not entirely inconceivable that some H-system, even though it is consistent, might not have any non-zero solutions $T : \mathbb{Z}^d \to \mathbb{C}$ at all. However, we prove in Section 5 that this is not the case.

As our proof of this fact is based on the well-known Ore-Sato structure theorem [5, 6, 8], we use this opportunity to remark in Section 6 that, contrary to some interpretations found in the literature (e.g., [3, 4]), the Ore-Sato theorem does *not* imply that every solution of an H-system is of the form

$$R(n_1,n_2,\ldots,n_d)\frac{\prod_{i=1}^{p}\Gamma(a_{i,1}n_1+n_2+\cdots+a_{i,d}n_d+\alpha_i)}{\prod_{j=1}^{q}\Gamma(b_{j,1}n_1+n_2+\cdots+b_{j,d}n_d+\beta_j)}\,u_1^{n_1}u_2^{n_2}\cdots u_d^{n_d} \qquad (1)$$

where $R \in \mathbb{C}(x_1,x_2,\ldots,x_d)$, $a_{ik}, b_{jk} \in \mathbb{Z}$, and $\alpha_i, \beta_j \in \mathbb{C}$ (for the case when the solution of the system is holonomic, and R is required to be a polynomial, we have already noted this in [2]). We conclude by giving an appropriate corollary to the Ore-Sato theorem on possible forms of solutions of systems under consideration.

We write $p \perp q$ to indicate that polynomials $p, q \in \mathbb{C}[x_1,x_2,\ldots,x_d]$ are relatively prime. We write $\mathbf{u} = (u_1,u_2,\ldots,u_d)$ for d-tuples of numbers or indeterminates, and $\mathbf{u}\cdot\mathbf{v} = \sum_{i=1}^{d} u_i v_i$ for their inner product. We denote by $\mathbf{e}_i$ the d-tuple whose components are zero except the i-th one which is 1. The monomial $x_1^{u_1}x_2^{u_2}\cdots x_d^{u_d}$ is denoted by $\mathbf{x}^{\mathbf{u}}$. A polynomial $p \in \mathbb{C}[\mathbf{x}]$ is *integer-linear* if $p(\mathbf{x}) = \mathbf{a}\cdot\mathbf{x}+\beta$ where $\mathbf{a} \in \mathbb{Z}^d$ and $\beta \in \mathbb{C}$. The set of positive integers is denoted by $\mathbb{N}$, and the set of non-negative integers by $\mathbb{N}_0$.

2 H-systems and their solution spaces

Definition 1 *An H-*system[1] *of dimension d is a system of equations of the form*

$$f_i(\mathbf{n})T(\mathbf{n}+\mathbf{e}_i) = g_i(\mathbf{n})T(\mathbf{n}) \tag{2}$$

for $i = 1, 2, \ldots, d$, where $f_i, g_i \in \mathbb{C}[\mathbf{x}] \setminus \{0\}$ and $f_i \perp g_i$. The rational functions $g_i/f_i \in \mathbb{C}(\mathbf{x})\setminus\{0\}$, $i = 1, 2, \ldots, d$, are called the certificates *of (2), and a function $T : \mathrm{dom}(T) \to \mathbb{C}$ is a* solution *of (2) if (2) is satisfied for all $\mathbf{n} \in \mathrm{dom}(T)$ such that $\mathbf{n}+\mathbf{e}_i \in \mathrm{dom}(T)$ for $i = 1, 2, \ldots, d$. A solution of an H-*system *is called a* hypergeometric term.

Definition 2 *Rational functions $F_1, F_2, \ldots, F_d \in \mathbb{C}(\mathbf{x}) \setminus \{0\}$ are* compatible *if*

$$F_i(\mathbf{x})F_j(\mathbf{x}+\mathbf{e}_i) \;=\; F_j(\mathbf{x})F_i(\mathbf{x}+\mathbf{e}_j)$$

for all $1 \le i < j \le d$. We call an H-system of the form (2) consistent *if its certificates are compatible.*

If an H-system has a solution with Zariski-dense[2] support, then it is consistent, and its certificates are uniquely determined by this solution (see [2]). Note that in the case $d = 1$, every H-system (containing a single equation) is consistent.

Definition 3 *Let $\mathcal{H}$ be an H-system of the form (2). A point $\mathbf{n} \in \mathbb{Z}^d$ is*

- *a* trailing integer singularity *of $\mathcal{H}$ if there exists i, $1 \le i \le d$, such that $g_i(\mathbf{n}) = 0$;*
- *a* leading integer singularity *of $\mathcal{H}$ if there exists i, $1 \le i \le d$, such that $f_i(\mathbf{n}-\mathbf{e}_i) = 0$;*
- *an* integer singularity *of $\mathcal{H}$ if it is a leading or a trailing integer singularity of $\mathcal{H}$.*

Definition 4 *Let $S(\mathcal{H})$ denote the set of all integer singularities of $\mathcal{H}$. Denote*

- *by $V_1(\mathcal{H})$ the $\mathbb{C}$-linear space of all solutions T of $\mathcal{H}$ such that $\mathrm{dom}\, T = \mathbb{Z}^d$, and*
- *by $V_2(\mathcal{H})$ the $\mathbb{C}$-linear space of all solutions T of $\mathcal{H}$ such that $\mathrm{dom}\, T = \mathbb{Z}^d \setminus S(\mathcal{H})$.*

[1] The prefix "H" refers to Jakob Horn and to the adjective "hypergeometric" as well.

[2] Recall that a set $S \subseteq \mathbb{C}^d$ is Zariski-dense if the only polynomial $p \in \mathbb{C}[x_1, x_2, \ldots, x_d]$ which vanishes at each point of S is the zero polynomial $p = 0$.

We consider only integer singularities here, therefore we will drop the adjective "integer" in the sequel. Sometimes we will also drop the name of the H-system, and will write V_1, V_2 instead of $V_1(\mathcal{H})$, $V_2(\mathcal{H})$.

Definition 5 *Two points* $\mathbf{p}, \mathbf{p}' \in \mathbb{Z}^d$ *are* adjacent *if* $\mathbf{p} - \mathbf{p}' = \pm \mathbf{e}_i$ *for some* $i \in \{1, 2, \ldots, d\}$. *A finite sequence* $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_k \in \mathbb{Z}^d$ *is a* path from $\mathbf{p}_1$ to $\mathbf{p}_k$ of length $k-1$ *if* $\mathbf{p}_i$ *is adjacent to* $\mathbf{p}_{i+1}$ *for* $i = 1, 2, \ldots, k-1$. *Given an H-system* $\mathcal{H}$, *the* components induced by $\mathcal{H}$ on $\mathbb{Z}^d$ *are the equivalence classes of the following equivalence relation* $\sim$ *in* $\mathbb{Z}^d$*:* $\mathbf{p}' \sim \mathbf{p}''$ *iff there exists a path from* $\mathbf{p}'$ *to* $\mathbf{p}''$ *which contains no singularity of* $\mathcal{H}$. *If* T *is a solution of* $\mathcal{H}$, *then for each component* C *induced by* $\mathcal{H}$ *on* $\mathbb{Z}^d$, *the restriction of* T *to* C *is called a* constituent *of* T.

Proposition 1 *Let* $\mathcal{H}$ *be a consistent H-system. Then* $\dim V_2$ *equals the number of components induced by* $\mathcal{H}$ *on* $\mathbb{Z}^d$.

Proof: To each component C_i induced by $\mathcal{H}$ on $\mathbb{Z}^d$ we assign a solution T_i of (2) which is 1 at a selected point $\mathbf{p}_i \in C_i$, and 0 on all the remaining components. The values of T_i on the remaining points of C_i are uniquely determined by (2). It is clear that the set of all T_i is a basis for V_2. □

3 The univariate case

When $d = 1$ the system (2) is of the form

$$f(n)T(n+1) = g(n)T(n) \tag{3}$$

where $f(n), g(n) \in \mathbb{C}[n] \setminus \{0\}$ and $f(n) \perp g(n)$.

Example 2 ($\dim V_1 = 1$, $\dim V_2 = k$) Consider the recurrence

$$T(n+1) = p_k(n)\ T(n) \tag{4}$$

where $k \geq 1$ and $p_k(n) = \prod_{i=0}^{k-2}(n - 2i + 1)$. Here we use the convention that a product is 1 if its lower limit exceeds its upper limit. Clearly the set of singularities of (4) is $\{2i-1\,;\ i = 0, 1, \ldots, k-2\}$, so $\dim V_2 = k$. To compute $\dim V_1$, note that any solution $T(n)$ of (4) defined for all $n \in \mathbb{Z}$ is a constant multiple of

$$F_k(n) = \begin{cases} (-1)^{(k-1)n} / \prod_{i=0}^{k-2}(2i - n - 1)!, & n < 0, \\ 0, & n \geq 0. \end{cases}$$

Therefore $\dim V_1 = 1$. ■

Example 3 ($\dim V_1 = m$, $\dim V_2 = m+1$) Now consider the recurrence

$$q_m(n+1)\ T(n+1) = q_m(n)\ T(n) \tag{5}$$

where $m \geq 1$ and $q_m(n) = \prod_{i=1}^{m}(n+2i+1)$. The set of singularities is $\{-(2i+1)\,;\ i = 1, 2, \ldots, m\}$, so $\dim V_2 = m+1$. Let $T(n)$ be a solution of (5) defined for all $n \in \mathbb{Z}$. By substituting $n = -2(i+1)$ for $i = 1, 2, \ldots, m$ into (5), we see that $T(n) = 0$ for these values of n. Likewise, by substituting $n = -3$ into (5), we find that $T(-2) = 0$. Using (5) it follows by induction on n that $T(n) = 0$ for all $n \leq -2(m+1)$ and for all $n \geq -2$ as well. On the other hand, it is easy to check that

$$G_m^{(i)}(n) = \delta_{n,-(2i+1)}$$

(where δ is the Kronecker delta) is a solution of (5) for $i = 1, 2, \ldots, m$. Therefore $\dim V_1 = m$. ■

Before describing the general situation we need a definition and a lemma.

Definition 6 *Let $\mathcal{H}$ be an H-system of the form (3). An interval of integers*

$$I = \{k, k+1, \ldots, k+m\}, \quad m \geq 0, \tag{6}$$

is a segment of singularities *of $\mathcal{H}$ if $I \subseteq S(\mathcal{H})$ while $k-1, k+m+1 \notin S(\mathcal{H})$.*

Lemma 1 *Each segment of singularities (6) of equation (3) is of (at least) one of the following types:*

(i) all elements of the segment are trailing singularities;

(ii) all elements of the segment are leading singularities;

(iii) there exists j, $0 \leq j < m$, such that $k, k+1, \ldots, k+j$ are leading singularities, while $k+j+1, k+j+2, \ldots, k+m$ are trailing singularities.

Proof: If $u \in \mathbb{Z}$ is a trailing singularity and $u+1$ a leading singularity of (3) then $f(u) = g(u) = 0$, contrary to the assumption $f \perp g$. So any segment of singularities of (3) consists of a (possibly empty) interval of leading singularities followed by a (possibly empty) interval of trailing singularities. □

Theorem 1 *Let S denote the set of singularities of equation (3).*
a) If $S = \emptyset$ then $\dim V_1 = \dim V_2 = 1$.
b) If $S \neq \emptyset$ then $1 \leq \dim V_1 < \dim V_2 < \infty$.

Proof: a) This is clear.

b) There is only a finite set of components induced on $\mathbb{Z}$ by (3), therefore $\dim V_2 < \infty$.

Next we prove that $\dim V_1 < \dim V_2$. First we show that if (6) is a segment of singularities of (3), then the restriction of V_1 to

$$\hat{I} = \{k-1, k, \ldots, k+m, k+m+1\}$$

has dimension ≤ 1, while the analogous restriction of V_2 obviously has dimension 2. Indeed, if u is a trailing singularity, then any element of V_1 vanishes at $u+1$; and if u is a leading singularity, then any element of V_1 vanishes at $u-1$. By Lemma 1 we have three possibilities (i), (ii), (iii) for (6). In case (i) we have $T(k+1) = T(k+2) = \ldots = T(k+m+1) = 0$, in case (ii) $T(k-1) = T(k) = \ldots = T(k+m-1) = 0$, in case (iii) $T(k-1) = T(k) = \ldots T(k+j-1) = 0$ and $T(k+j+2) = T(k+j+3) = \ldots = T(k+m+1) = 0$; in each case $T(n)$ can be non-zero at most in two points of $\hat{I}$, however the value at one of them is uniquely determined by the value at the other one. Therefore the dimension of the restricted V_1 is ≤ 1. The same holds for dimension of the restriction of V_1 to the set

$$\{k-v, k-v+1, \ldots, k, k+1, \ldots, k+m, k+m+1, \ldots, k+w\},$$

where $k, k+1, \ldots, k+m$ are singularities, while $k-v, k-v+1, \ldots, k-1$ and $k+m+1, k+m+2, \ldots, k+w$ are not. Gluing together two such restrictions with coinciding, say, $k+m+1, k+m+2, \ldots, k+w$, and non-intersecting singular parts, we get the dimension ≤ 2, while the dimension of the corresponding restriction of V_2 is 3 and so on. This proves that $\dim V_1 < \dim V_2$.

Finally we prove that $\dim V_1 \geq 1$. If there are leading singularities, let n_0 be the largest leading singularity. Set $T(n_0) = 1$ and $T(n) = 0$ for $n < n_0$. None of the points $n > n_0$ is a leading singularity, hence the value of T at $n > n_0$ is uniquely determined by the recurrence (3) and the initial condition $T(n_0) = 1$. If there are no leading singularities, let n_0 be the least trailing singularity. Set $T(n_0) = 1$ and $T(n) = 0$ for $n > n_0$. None of the points $n < n_0$ is a trailing singularity, hence the value of T at $n < n_0$ is uniquely determined by the recurrence (3) and the initial condition $T(n_0) = 1$. In either case V_1 contains a non-zero solution. $\square$

Theorem 2 *For any integers s, t such that $1 \leq s < t$ there exists an equation of the form (3) such that $\dim V_1 = s$ and $\dim V_2 = t$.*

Proof: Consider the recurrence

$$q_m(n+1)\ T(n+1) = p_k(n) q_m(n)\ T(n) \tag{7}$$

where $k, m \geq 1$, $p_k(n)$ is as in Example 2, and $q_m(n)$ is as in Example 3. Here the set of singularities is $\{2i-1\,;\ i = 0, 1, \ldots, k-2\} \cup \{-(2i+1);\ i = 1, 2, \ldots, m\}$, so $\dim V_2 = k+m$. Let $T(n)$ be a solution of (7) defined for all $n \in \mathbb{Z}$. In exactly the same way as in Example 3 we can see that $T(n) = 0$ for $n =$

$-2, -4, \ldots, -2(m+1)$, $n \leq -2(m+1)$ or $n \geq -2$, and that $G_m^{(i)}(n) = \delta_{n,-(2i+1)}$ is a solution of (7) for $i = 1, 2, \ldots, m$. Therefore $\dim V_1 = m$.

If $1 \leq s < t$, let $m = s$ and $k = t - s$. Then for equation (7), $\dim V_1 = m = s$ and $\dim V_2 = k + m = t$. □

We conclude this section by some remarks on computation of $\dim V_1$ and $\dim V_2$. Let $\mathcal{H}$ denote equation (3). According to Proposition 1, $\dim V_2(\mathcal{H})$ is the number of components induced on $\mathbb{Z}$ by $\mathcal{H}$ and is thus easy to compute. We claim that $\dim V_1(\mathcal{H})$ equals the dimension of the kernel of a bidiagonal matrix B defined as follows. Let α be the maximum and β the minimum of the integer roots of $f(x)g(x)$; if $\mathcal{H}$ has no integer singularities then we can take $\alpha = \beta = 1$. Let B be the $(\alpha - \beta + 1) \times (\alpha - \beta + 2)$ matrix with entries

$$b_{i,j} = \begin{cases} f(\alpha - i + 1), & j = i, \\ -g(\alpha - i + 1), & j = i + 1, \\ 0, & \text{otherwise}, \end{cases}$$

where $1 \leq i \leq \alpha - \beta + 1$ and $1 \leq j \leq \alpha - \beta + 2$. Indeed, any vector v such that $Bv = 0$ can be extended to a solution of $\mathcal{H}$ in a unique way. This mapping is an isomorphism between the kernel of B and $V_1(\mathcal{H})$.

Incidentally, this gives an alternative proof of the inequality $\dim V_1 \geq 1$: B has more columns than rows, hence its kernel is nontrivial.

4 The relation between dimensions of V_1 and V_2 in the multivariate case

If $d \geq 2$ in (2) then the dimensions of V_1 and/or V_2 can be infinite as shown by the following examples.

Example 4 ($\dim V_1 = \infty$, $\dim V_2 = 1$) Let $\mathcal{H}$ be the system

$$\begin{aligned} (n_1 - 4n_2 + 1)T(n_1 + 1, n_2) &= (n_1 - 4n_2)T(n_1, n_2), \\ (n_1 - 4n_2 - 4)T(n_1, n_2 + 1) &= (n_1 - 4n_2)T(n_1, n_2). \end{aligned}$$

It is easy to check that

$$T_i(n_1, n_2) = \delta_{n_1,4i}\delta_{n_2,i}, \qquad \text{for } i \in \mathbb{Z},$$

are linearly independent solutions of $\mathcal{H}$ on all of $\mathbb{Z}^2$, hence $\dim V_1 = \infty$. On the other hand, $S(\mathcal{H}) = \{(n_1, n_2)\,;\ n_1 = 4n_2\}$, so $\mathcal{H}$ induces a single component on $\mathbb{Z}^2$, and $\dim V_2 = 1$. ■

Example 5 ($\dim V_1 = 1$, $\dim V_2 = \infty$) Let $\mathcal{B}$ be the system

$$\begin{aligned} (n_1 - 4n_2)T(n_1 + 1, n_2) &= (n_1 - 4n_2 + 1)T(n_1, n_2), \\ (n_1 - 4n_2)T(n_1, n_2 + 1) &= (n_1 - 4n_2 - 4)T(n_1, n_2). \end{aligned}$$

It can be shown that any solution of $\mathcal{B}$ defined on all $\mathbb{Z}^2$ is a constant multiple of $n_1 - 4n_2$, so $\dim V_1 = 1$. On the other hand, $S(\mathcal{B}) = \{(n_1, n_2)\,;\; n_1 - 4n_2 \in \{-4, -1, 1, 4\}\}$, so each of the points $(4i, i)$ for $i \in \mathbb{Z}$ is a separate component of $\mathbb{Z}^2$ induced by $\mathcal{B}$, hence $\dim V_2 = \infty$. ∎

Example 6 ($\dim V_1 = \dim V_2 = \infty$) Let $\mathcal{C}$ be the system

$$\begin{aligned}(n_1 - n_2 - 1)(n_1 - n_2 + 1)T(n_1 + 1, n_2) &= (n_1 - n_2)(n_1 - n_2 + 2)T(n_1, n_2),\\ (n_1 - n_2 - 1)(n_1 - n_2 + 1)T(n_1, n_2 + 1) &= (n_1 - n_2)(n_1 - n_2 - 2)T(n_1, n_2).\end{aligned}$$

It is easy to check that

$$T_i(n_1, n_2) = \delta_{n_1,i}\delta_{n_2,i}, \qquad \text{for } i \in \mathbb{Z}, \tag{8}$$

are linearly independent solutions of $\mathcal{C}$ on all of $\mathbb{Z}^2$, hence $\dim V_1 = \infty$. As $S(\mathcal{C}) = \{(n_1, n_2)\,;\; n_1 - n_2 \in \{-2, 0, 2\}\}$, each of the points $(i, i-1)$ and $(i, i+1)$ for $i \in \mathbb{Z}$ is a separate component of $\mathbb{Z}^2$ induced by $\mathcal{C}$, so $\dim V_2 = \infty$ as well. ∎

The following theorem describes the general situation.

Theorem 3 *Let $1 \le s, t \le \infty$. Then there exists an H-system such that* $\dim V_1 = s$ *and* $\dim V_2 = t$.

Proof: Let $t \ge 2$ and $p_t(n_1, n_2) = \prod_{i=0}^{t-2}(n_1 - n_2 + 3i)$. Then the set of singularities of

$$\begin{aligned}p_t(n_1 + 1, n_2)T(n_1 + 1, n_2) &= p_t(n_1, n_2)T(n_1, n_2),\\ p_t(n_1, n_2 + 1)T(n_1, n_2 + 1) &= p_t(n_1, n_2)T(n_1, n_2)\end{aligned}$$

is $S = \{(n_1, n_2)\,;\; n_1 - n_2 \in \{-3i\,;\; 0 \le i \le t - 2\}\}$. As in Example 6, the functions (8) are linearly independent solutions of this system on all of $\mathbb{Z}^2$, hence $\dim V_1 = \infty$. On the other hand, the number of components induced on $\mathbb{Z}^2$ is t, so $\dim V_2 = t$.

Let $s \ge 2$ and

$$q_s(n_1, n_2) = \prod_{i=1}^{s-1}((n_1 - 2i)^2 + n_2^2). \tag{9}$$

Then the set of singularities of

$$\begin{aligned}(n_1 - 4n_2)q_{s+1}(n_1 + 1, n_2)T(n_1 + 1, n_2) &= (n_1 - 4n_2 + 1)q_{s+1}(n_1, n_2)T(n_1, n_2),\\ (n_1 - 4n_2)q_{s+1}(n_1, n_2 + 1)T(n_1, n_2 + 1) &= (n_1 - 4n_2 - 4)q_{s+1}(n_1, n_2)T(n_1, n_2)\end{aligned}$$

is $S = \{(n_1, n_2)\,;\; n_1 - 4n_2 \in \{-4, -1, 1, 4\}\} \cup \{(2i, 0)\,;\; 1 \le i \le s\}$. Each of the points $(4i, i)$ for $i \in \mathbb{Z}$ is a separate component, so $\dim V_2 = \infty$. It can be shown that any solution $T(n_1,n_2)$ defined on all $\mathbb{Z}^2$ vanishes everywhere except at the points $(2i,0)$ where $1 \le i \le s$, and that

$$T_i(n_1,n_2) = \delta_{n_1,2i}\delta_{n_2,0}, \tag{10}$$

for $i = 1,2,\ldots,s$, are linearly independent solutions of this system defined on all $\mathbb{Z}^2$. Hence $\dim V_1 = \infty$.

Together with Examples 4 – 6 this proves the assertion in the case when at least one of s, t is infinite.

Now assume that s, t are natural numbers, and let $r_t(n_1,n_2) = \prod_{i=1}^{t-1}(n_1 + 2i+1)$. Consider the system

$$\begin{aligned} q_s(n_1+1,n_2)T(n_1+1,n_2) &= q_s(n_1,n_2)r_t(n_1,n_2)T(n_1,n_2), \\ q_s(n_1,n_2+1)T(n_1,n_2+1) &= q_s(n_1,n_2)T(n_1,n_2), \end{aligned}$$

where q_s is as in (9). It can be shown that any solution $T(n_1,n_2)$ defined on all $\mathbb{Z}^2$ vanishes for all (n_1,n_2) such that $n_1 > -(2t-1)$ and (n_1,n_2) is not of the form $(2i,0)$ with $1 \le i \le s-1$. Further, a basis of V_1 is given by the s functions $T_i(n_1,n_2)$ for $i = 0,1,\ldots,s-1$ where

$$T_0(n_1,n_2) = \begin{cases} \frac{(-1)^{(t-1)n_1}}{\prod_{i=1}^{s-1}((n_1-2i)^2+n_2^2)\prod_{i=1}^{t-1}(-n_1-2i-1)!}, & n_1 \le -(2t-1), \\ 0, & \text{otherwise}, \end{cases}$$

and $T_i(n_1,n_2)$ are as in (10) for $i = 1,2,\ldots,s-1$. It follows that $\dim V_1 = s$. The set of singularities of this system is $S = \{(2i,0)\,;\ 1 \le i \le s-1\} \cup \{(-(2i+1),j)\,;\ 1 \le i \le t-1, j \in \mathbb{Z}\}$, and the number of components induced on $\mathbb{Z}^2$ is t, so $\dim V_2 = t$ as desired. $\square$

We considered the case $d = 2$ here. The corresponding *H*-systems for the case of an arbitrary $d > 1$ can be obtained by adding equations $T(\mathbf{n} + \mathbf{e}_i) = T(\mathbf{n})$, $i = 3,4,\ldots,d$, to the systems with $d = 2$.

5 Existence of solutions in the multivariate case

In this section we assume that $\mathcal{H}$ is a consistent *H*-system of the form (2), and show that $\dim V_1(\mathcal{H}) > 0$.

At first glance, it seems that obtaining a non-zero solution T of $\mathcal{H}$, defined everywhere on $\mathbb{Z}^d$, is trivial: Select any point $\mathbf{s} \in \mathbb{Z}^d$ and define $T(\mathbf{s}) = 1$, then extend T to all of $\mathbb{Z}^d$ by recursion using $\mathcal{H}$. However, if $S(\mathcal{H}) \neq \emptyset$ this simple idea may fail: for instance, in Example 1 the only possible starting point is $\mathbf{s} = (0,0)$. Since this is a singularity of $\mathcal{H}$, we refine the idea by always selecting $\mathbf{s} \in S(\mathcal{H})$. That this, too, can fail, is shown by the following system.

Example 7 Let $\mathcal{H}$ be the consistent H-system

$$\begin{aligned}(n_1-n_2+2)((n_1+1)^2+n_2^2)T(n_1+1,n_2) &= (n_1-n_2)(n_1^2+n_2^2)T(n_1,n_2),\\ (n_1-n_2-1)(n_1^2+(n_2+1)^2)T(n_1,n_2+1) &= (n_1-n_2+1)(n_1^2+n_2^2)T(n_1,n_2)\end{aligned}$$

with $S(\mathcal{H}) = \{(n_1,n_2) \in \mathbb{Z}^2;\ (n_1-n_2)(n_1-n_2+1)(n_1^2+n_2^2) = 0\}$. It is easy to check that, as in Example 1, $V_1(\mathcal{H})$ is spanned by

$$T(n_1,n_2) = \begin{cases} 1, & n_1 = n_2 = 0,\\ 0, & \text{otherwise.}\end{cases}$$

Selecting $\mathbf{s} = (0,0)$ and defining $T(\mathbf{s}) = 1$ will indeed produce the non-zero solution $T(n_1,n_2) = \delta_{n_1,0}\delta_{n_2,0}$. However, as every element of $V_1(\mathcal{H})$ vanishes at all $\mathbf{n} \neq (0,0)$, any other choice of $\mathbf{s}$, including all the other singular points of $\mathcal{H}$, will lead to contradiction. ■

In general, it is not clear how to select $\mathbf{s}$, or even if a "good" $\mathbf{s}$ exists at all. We will now show that it does.

A sketch of the route to be taken is the following. To each rational function $R(\mathbf{x})$ we will associate the sequence of rational functions $\hat{R}(\mathbf{n}) := R(\mathbf{n}+\mathbf{x})$. Using the Ore-Sato theorem (Theorem 4), we will construct a sequence of rational functions $\varphi : \mathbb{Z}^d \to \mathbb{C}(\mathbf{x})$ which solves the modified H-system

$$\widehat{f_i}(\mathbf{n})\varphi(\mathbf{n}+\mathbf{e}_i) = \widehat{g_i}(\mathbf{n})\varphi(\mathbf{n}),\quad i = 1,2,\ldots,d$$

over $\mathbb{C}(\mathbf{x})$. We will define an integer valuation $\operatorname{val} R(\mathbf{x})$ for any $R(\mathbf{x}) \in \mathbb{C}(\mathbf{x})$. The key point of our proof will be the fact that the sequence $\operatorname{val}\varphi(\mathbf{n})$ is bounded (Proposition 4), which will enable us to associate with $\mathcal{H}$ the set $M_{\mathcal{H}} = \{\mathbf{n} \in \mathbb{Z}^d\ ;\ \operatorname{val}\varphi(\mathbf{n}) = m\}$ where $m = \min_{\mathbf{n}\in\mathbb{Z}^d}\operatorname{val}\varphi(\mathbf{n})$. Then we will prove that for any $\mathbf{s} \in M_{\mathcal{H}}$ we can construct a solution $T(\mathbf{n}) \in V_1(\mathcal{H})$ such that $T(\mathbf{s}) = 1$.

Let K be a field. For $k \in \mathbb{Z}$ and $\alpha \in K$, denote by $\wp(\alpha;k)$ the *Pochhammer symbol*

$$\wp(\alpha;k) = \begin{cases} \displaystyle\prod_{j=0}^{k-1}(\alpha+j), & k \geq 0,\\ \displaystyle\prod_{j=1}^{|k|}\frac{1}{\alpha-j}, & k < 0,\quad \alpha \neq 1,2,\ldots,|k|.\end{cases}$$

Theorem 4 (Ore-Sato) *Let $\{G_{\mathbf{n}}(\mathbf{x}) \in \mathbb{C}(\mathbf{x})\ ;\ \mathbf{n} \in \mathbb{Z}^d\}$ be a family of rational functions satisfying the* cocycle condition

$$\forall\, \mathbf{n},\mathbf{m} \in \mathbb{Z}^d : \quad G_{\mathbf{n}+\mathbf{m}}(\mathbf{x}) = G_{\mathbf{n}}(\mathbf{x}) \cdot G_{\mathbf{m}}(\mathbf{x}+\mathbf{n}). \tag{11}$$

Then we can write

$$G_{\mathbf{n}}(\mathbf{x}) = C(\mathbf{n}) \cdot \prod_{j=1}^{p} \wp(\mathbf{a}^{(j)}\cdot\mathbf{x}+\beta_j;\ \mathbf{a}^{(j)}\cdot\mathbf{n})^{s_j} \cdot \frac{R(\mathbf{x}+\mathbf{n})}{R(\mathbf{x})} \tag{12}$$

where $C:\mathbb{Z}^d \to \mathbb{C}$ *satisfies* $C(\mathbf{n}+\mathbf{m}) = C(\mathbf{n})C(\mathbf{m})$, $p \in \mathbb{N}_0$, $\mathbf{a}^{(j)} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}$, $\beta_j \in \mathbb{C}$, $s_j \in \mathbb{Z} \setminus \{0\}$, *and* $R(\mathbf{x}) \in \mathbb{C}(\mathbf{x})$.

For a proof, see [8, pp. 26–33][3].

Corollary 1 *Let* $F_1(\mathbf{x}), F_2(\mathbf{x}), \ldots, F_d(\mathbf{x}) \in \mathbb{C}(\mathbf{x})$ *be compatible rational functions (see Def. 2). Then for* $i = 1, 2, \ldots, d$ *we can write*

$$F_i(\mathbf{x}) = c_i \cdot \prod_{j=1}^{p} \wp(\mathbf{a}^{(j)} \cdot \mathbf{x} + \beta_j;\, a_i^{(j)})^{s_j} \cdot \frac{R(\mathbf{x}+\mathbf{e}_i)}{R(\mathbf{x})} \tag{13}$$

where $c_i \in \mathbb{C}$, $p \in \mathbb{N}_0$, $\mathbf{a}^{(j)} = (a_1^{(j)}, a_2^{(j)}, \ldots, a_d^{(j)}) \in \mathbb{Z}^d \setminus \{\mathbf{0}\}$, $\beta_j \in \mathbb{C}$, $s_j \in \mathbb{Z}\setminus\{0\}$, $R(\mathbf{x}) \in \mathbb{C}(\mathbf{x})\setminus\{0\}$, *the complete factorization of the numerator and denominator of* $R(\mathbf{x})$ *contains no integer-linear factors,* $\gcd(a_1^{(j)}, a_2^{(j)}, \ldots, a_d^{(j)}) = 1$, *and the first non-zero component of* $\mathbf{a}^{(j)}$ *is positive, for* $j = 1, 2, \ldots, p$.

Proof: Write $B = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_d\}$. To each sequence of unit vectors $\mathbf{d}_1, \mathbf{d}_2, \ldots, \mathbf{d}_r$ from $B \cup (-B)$ assign the rational function

$$\tilde{G}_{\mathbf{d}_1,\mathbf{d}_2,\ldots,\mathbf{d}_r}(\mathbf{x}) = \prod_{j=1}^{r} \tilde{F}_{\mathbf{d}_j}(\mathbf{x} + \mathbf{n}_{j-1})$$

where

$$\tilde{F}_{\mathbf{d}}(\mathbf{x}) = \begin{cases} F_i(\mathbf{x}), & \mathbf{d} = \mathbf{e}_i, \\ F_i(\mathbf{x}-\mathbf{e}_i)^{-1}, & \mathbf{d} = -\mathbf{e}_i, \end{cases}$$

and $\mathbf{n}_j = \sum_{i=1}^{j} \mathbf{d}_i$, for $0 \le j \le r$. As $F_1, F_2, \ldots, F_d$ are compatible, $\tilde{G}_{\mathbf{d}_1,\mathbf{d}_2,\ldots,\mathbf{d}_r}(\mathbf{x})$ does not change if two consecutive terms in the sequence $\mathbf{d}_1, \mathbf{d}_2, \ldots, \mathbf{d}_r$ are transposed. Hence $\tilde{G}_{\mathbf{d}_1,\mathbf{d}_2,\ldots,\mathbf{d}_r}(\mathbf{x}) = \tilde{G}_{\mathbf{d}_{\pi(1)},\mathbf{d}_{\pi(2)},\ldots,\mathbf{d}_{\pi(r)}}(\mathbf{x})$ for any permutation π of $\{1, 2, \ldots, r\}$. In particular, we can sort the sequence $\mathbf{d}_1, \mathbf{d}_2, \ldots, \mathbf{d}_r$ into a sequence of the form $\mathbf{e}_1, \ldots, \mathbf{e}_1, -\mathbf{e}_1, \ldots, -\mathbf{e}_1, \ldots, \mathbf{e}_d, \ldots, \mathbf{e}_d, -\mathbf{e}_d, \ldots, -\mathbf{e}_d$. Since $\tilde{G}_{\mathbf{d}_i,\mathbf{d}_{i+1}}(\mathbf{x}) = 1$ if $\mathbf{d}_i = -\mathbf{d}_{i+1}$, by definition of $\tilde{F}_{\mathbf{d}}$, and

$$\tilde{G}_{\mathbf{d}_1,\mathbf{d}_2,\ldots,\mathbf{d}_r,\mathbf{d}_{r+1},\ldots,\mathbf{d}_s}(\mathbf{x}) = \tilde{G}_{\mathbf{d}_1,\mathbf{d}_2,\ldots,\mathbf{d}_r}(\mathbf{x}) \cdot \tilde{G}_{\mathbf{d}_{r+1},\ldots,\mathbf{d}_s}(\mathbf{x}+\mathbf{n}_r), \tag{14}$$

by definition of $\tilde{G}$, this sequence can be reduced by omitting each consecutive pair of $\mathbf{e}_i$ and $-\mathbf{e}_i$. It follows that $\tilde{G}_{\mathbf{d}_1,\mathbf{d}_2,\ldots,\mathbf{d}_r}$ depends only on $\mathbf{d}_1 + \mathbf{d}_2 + \cdots + \mathbf{d}_r = \mathbf{n}_r$.

[3] In fact, a more general version of the Ore-Sato theorem is proved in [8], with $\mathbb{Z}^d$ replaced by an arbitrary abelian group Ξ generated by d elements, and with $\mathbb{C}$ replaced by an arbitrary algebraically closed field Ω of characteristic zero. Note however that in the statement and proof of this theorem in [8], $\prod_{l \le k \le -1} \psi_i(x+k)^{-1}$ should be replaced by $\prod_{l \le k \le -1} \psi_i(x-k)^{-1}$.

Thus we can define a family of rational functions $\{G_{\mathbf{n}}(\mathbf{x}) \in \mathbb{C}(\mathbf{x})\,;\ \mathbf{n} \in \mathbb{Z}^d\}$ by setting

$$G_{\mathbf{n}}(\mathbf{x}) = \tilde{G}_{\mathbf{d}_1,\mathbf{d}_2,\ldots,\mathbf{d}_r}(\mathbf{x})$$

where $\mathbf{d}_1, \mathbf{d}_2, \ldots, \mathbf{d}_r$ is any sequence of vectors from $B \cup (-B)$ summing to $\mathbf{n}$. Because of (14), the family $\{G_{\mathbf{n}}(\mathbf{x})\,;\ \mathbf{n} \in \mathbb{Z}^d\}$ satisfies the cocycle condition (11), hence by Theorem 4, $G_{\mathbf{n}}(\mathbf{x})$ has the form (12). Notice that $G_{\mathbf{e}_i}(\mathbf{x}) = F_i(\mathbf{x})$ and $\mathbf{a}^{(j)} \cdot \mathbf{e}_i = a_i^{(j)}$, so with $\mathbf{n} = \mathbf{e}_i$ and $C(\mathbf{e}_i) = c_i$, (12) turns into (13).

If $R(\mathbf{x}) = \tilde{R}(\mathbf{x})\,(\mathbf{a} \cdot \mathbf{x} + \beta)^s$ where $\tilde{R}(\mathbf{x}) \in \mathbb{C}(\mathbf{x})$, $\mathbf{a} \in \mathbb{Z}^d$, $\beta \in \mathbb{C}$ and $s \in \mathbb{Z}$, then $R(\mathbf{x}+\mathbf{e}_i)/R(\mathbf{x}) = \tilde{R}(\mathbf{x}+\mathbf{e}_i)/\tilde{R}(\mathbf{x}) \cdot \wp(\mathbf{a}\cdot\mathbf{x}+\beta+1;\, a_i)^s/\wp(\mathbf{a}\cdot\mathbf{x}+\beta;\, a_i)^s$. Thus we can extract all integer-linear factors from R and replace them by appropriate Pochhammer symbols in the product in (13).

The last two claims follow from the formulæ

$$\wp(\mathbf{a} \cdot \mathbf{x} + \beta;\, a_i) \;=\; \delta^{a_i} \prod_{k=0}^{\delta-1} \wp((\mathbf{a} \cdot \mathbf{x} + \beta + k)/\delta;\, a_i/\delta)$$

where $\delta = \gcd(a_1, a_2, \ldots, a_d)$, and

$$\wp(\mathbf{a} \cdot \mathbf{x} + \beta;\, a_i)\wp(1 - \mathbf{a} \cdot \mathbf{x} - \beta;\, -a_i) \;=\; (-1)^{a_i},$$

both easily verifiable by direct computation. $\square$

To each rational function $R(\mathbf{x}) \in \mathbb{C}(\mathbf{x})$ we associate a sequence of rational functions $\hat{R} : \mathbb{Z}^d \to \mathbb{C}(\mathbf{x})$ by setting $\hat{R}(\mathbf{n}) = R(\mathbf{n} + \mathbf{x})$. Obviously we have

Proposition 2 *If $R(\mathbf{x})$ is not identically zero, then for all $\mathbf{n} \in \mathbb{Z}^d$, $\hat{R}(\mathbf{n})$ is not identically zero.*

Define a *valuation* $\mathrm{val} : \mathbb{C}(\mathbf{x}) \to \mathbb{Z}$ in the following way: For $p(\mathbf{x}) \in \mathbb{C}[\mathbf{x}]$ let

$$\mathrm{val}\, p \;=\; \min\{e_1 + e_2 + \cdots + e_d\,;\ x_1^{e_1} x_2^{e_2} \cdots x_d^{e_d} \text{ is a monomial of } p\}.$$

If $R(\mathbf{x}) \in \mathbb{C}(\mathbf{x}) \setminus \mathbb{C}[\mathbf{x}]$ and $R = p/q$ where $p, q \in \mathbb{C}[\mathbf{x}]$ and $p \perp q$, let

$$\mathrm{val}\, R = \mathrm{val}\, p - \mathrm{val}\, q.$$

Proposition 3 *(i) If $p \in \mathbb{C}[\mathbf{x}]$ then $\sum_{i=1}^d \deg_{x_i} p \geq \mathrm{val}\, p \geq 0$, and*

$$\mathrm{val}\, p > 0 \iff p(\mathbf{0}) = 0.$$

(ii) If $R_1, R_2 \in \mathbb{C}(\mathbf{x})$ then $\mathrm{val}\, R_1 R_2 = \mathrm{val}\, R_1 + \mathrm{val}\, R_2$.

Proof: Assertion (i) is obvious, and so is (ii) when $R_1, R_2 \in \mathbb{C}[\mathbf{x}]$. To prove (ii) in general, write $R_i = p_i/q_i$ where $p_i, q_i \in \mathbb{C}[\mathbf{x}]$ and $p_i \perp q_i$, for $i = 1, 2$. Denote $r = \gcd(p_1, q_2)$, $s = \gcd(p_2, q_1)$, $p_1' = p_1/r$, $q_2' = q_2/r$, $p_2' = p_2/s$, $q_1' = q_1/s$. Then $R_1 R_2 = p_1 p_2/(q_1 q_2) = p_1' p_2'/(q_1' q_2')$ where $p_1' p_2' \perp q_1' q_2'$. Hence $\mathrm{val}\, R_1 R_2 = \mathrm{val}\, p_1' p_2' - \mathrm{val}\, q_1' q_2' = \mathrm{val}\, p_1' + \mathrm{val}\, p_2' - \mathrm{val}\, q_1' - \mathrm{val}\, q_2' = \mathrm{val}\, p_1' + \mathrm{val}\, r + \mathrm{val}\, p_2' + \mathrm{val}\, s - \mathrm{val}\, q_1' - \mathrm{val}\, s - \mathrm{val}\, q_2' - \mathrm{val}\, r = \mathrm{val}\, p_1' r + \mathrm{val}\, p_2' s - \mathrm{val}\, q_1' s - \mathrm{val}\, q_2' r = \mathrm{val}\, p_1 + \mathrm{val}\, p_2 - \mathrm{val}\, q_1 - \mathrm{val}\, q_2 = \mathrm{val}\, R_1 + \mathrm{val}\, R_2$, as claimed. $\square$

Proposition 4 *Let $R \in \mathbb{C}(\mathbf{x})$. Then the sequence* $\operatorname{val} \hat{R}(\mathbf{n})$ *is bounded everywhere on $\mathbb{Z}^d$.*

Proof: Let $R = p/q$ where $p, q \in \mathbb{C}[\mathbf{x}]$ and $p \perp q$. By Proposition 3(i) we have

$$\begin{aligned} \sum_{i=1}^{d} \deg_{x_i} p &= \sum_{i=1}^{d} \deg_{x_i} \hat{p}(\mathbf{n}) \\ &\geq \operatorname{val} \hat{p}(\mathbf{n}) \geq \operatorname{val} \hat{R}(\mathbf{n}) \geq -\operatorname{val} \hat{q}(\mathbf{n}) \\ &\geq -\sum_{i=1}^{d} \deg_{x_i} \hat{q}(\mathbf{n}) \\ &= -\sum_{i=1}^{d} \deg_{x_i} q \end{aligned}$$

for any $\mathbf{n} \in \mathbb{Z}^d$. □

The following proposition is obvious.

Proposition 5 *If $F_1, F_2, \ldots, F_d \in \mathbb{C}(\mathbf{x})$ are compatible rational functions (see Def. 2), then the sequences $\widehat{F_1}, \widehat{F_2}, \ldots, \widehat{F_d}$ are also compatible in the sense that for any $\mathbf{n} \in \mathbb{Z}^d$, the rational functions $\widehat{F_i}(\mathbf{n})\widehat{F_j}(\mathbf{n}+\mathbf{e}_i)$ and $\widehat{F_j}(\mathbf{n})\widehat{F_i}(\mathbf{n}+\mathbf{e}_j)$ are equal in $\mathbb{C}(\mathbf{x})$, for $1 \leq i < j \leq d$.*

Let $\mathcal{H}$ be a consistent *H*-system of the form (2). By Corollary 1 we can write its certificates $F_i = g_i/f_i$ in the form (13). For $i = 1, 2, \ldots, d$, define

$$F_i'(\mathbf{x}) = c_i \cdot \prod_{j=1}^{p} \wp(\mathbf{a}^{(j)} \cdot \mathbf{x} + \beta_j;\, a_i^{(j)})^{s_j}, \tag{15}$$

$$F_i''(\mathbf{x}) = \frac{R(\mathbf{x} + \mathbf{e}_i)}{R(\mathbf{x})}. \tag{16}$$

Then $F_i = F_i' F_i''$ for $i = 1, 2, \ldots, d$. Since $F_1, F_2, \ldots, F_d$ as well as $F_1'', F_2'', \ldots, F_d''$ are compatible, so are $F_1', F_2', \ldots, F_d'$.

We will associate with $\mathcal{H}$ three sequences $\xi, \eta, \varphi : \mathbb{Z}^d \to \mathbb{C}(\mathbf{x})$ with rational-function values, defined by the following requirements:

- $\xi(\mathbf{0}) = 1$, $\xi(\mathbf{n} + \mathbf{e}_i) = \xi(\mathbf{n})\widehat{F_i'}(\mathbf{n})$, $i = 1, 2, \ldots, d$,
- $\eta(\mathbf{n}) = \hat{R}(\mathbf{n})$,
- $\varphi(\mathbf{n}) = \xi(\mathbf{n})\eta(\mathbf{n})$.

Notice that the existence and uniqueness of ξ follow from the compatibility of $\widehat{F_1'}, \widehat{F_2'}, \ldots, \widehat{F_d'}$. For $i = 1, 2, \ldots, d$ set

$$F_i' = \frac{g_i'}{f_i'},\ F_i'' = \frac{g_i''}{f_i''},$$

where $g_i', f_i', g_i'', f_i'' \in \mathbb{C}[\mathbf{x}]$, $g_i' \perp f_i'$, $g_i'' \perp f_i''$. Then ξ, η satisfy the systems

$$\widehat{f_i'}(\mathbf{n})\xi(\mathbf{n}+\mathbf{e}_i) = \widehat{g_i'}(\mathbf{n})\xi(\mathbf{n}),\quad i = 1, 2, \ldots, d, \tag{17}$$

$$\widehat{f_i''}(\mathbf{n})\eta(\mathbf{n}+\mathbf{e}_i) = \widehat{g_i''}(\mathbf{n})\eta(\mathbf{n}),\quad i = 1, 2, \ldots, d. \tag{18}$$

Since $R(\mathbf{x})$ contains no integer linear factors, no cancellation occurs on the left-hand side of

$$\frac{f_i' f_i''}{g_i' g_i''} = \frac{f_i}{g_i}.$$

Therefore $f_i' f_i'' = f_i$ and $g_i' g_i'' = g_i$, hence $\widehat{f_i'}\widehat{f_i''} = \widehat{f_i}$ and $\widehat{g_i'}\widehat{g_i''} = \widehat{g_i}$ as well. As a consequence of equalities (17), (18) we have

$$\widehat{f_i}(\mathbf{n})\varphi(\mathbf{n}+\mathbf{e}_i) = \widehat{g_i}(\mathbf{n})\varphi(\mathbf{n}),\quad i = 1, 2, \ldots, d. \tag{19}$$

Our next goal is to show that the sequence $\operatorname{val}\varphi(\mathbf{n})$ is bounded.

With any factor $\wp(\mathbf{a}^{(j)} \cdot \mathbf{x} + \beta_j;\, a_i^{(j)})$ in (15), we associate $|a_i^{(j)}|$ hyperplanes in $\mathbb{C}^d$: those hyperplanes are defined by the equations

$$\mathbf{a}^{(j)} \cdot \mathbf{x} + \beta_j + l = 0,\ l = 0, 1, \ldots, a_i^{(j)} - 1$$

if $a_i^{(j)} > 0$, and by

$$\mathbf{a}^{(j)} \cdot \mathbf{x} + \beta_j + l = 0,\ l = -1, -2, \ldots, a_i^{(j)}$$

if $a_i^{(j)} < 0$. All the factors from (15) generate a finite set of hyperplanes which we will denote by $\mathcal{P}$. The number of elements of $\mathcal{P}$ will be denoted by N. We call a point $\mathbf{n} \in \mathbb{Z}^d$ *special* if it belongs to at least one hyperplane from $\mathcal{P}$.

Proposition 6 *If two points $\mathbf{n}, \mathbf{n}' \in \mathbb{Z}^d$ are adjacent and $\operatorname{val}\xi(\mathbf{n}) \neq \operatorname{val}\xi(\mathbf{n}')$, then at least one of these points is special. In this case $|\operatorname{val}\xi(\mathbf{n}) - \operatorname{val}\xi(\mathbf{n}')| \leq |s_1| + |s_2| + \cdots + |s_p|$, where $s_1, s_2, \ldots, s_p$ are as in (15).*

Proof: From the definition of $\xi(\mathbf{n})$ and from Proposition 3(ii) it follows that $|\operatorname{val}\xi(\mathbf{n}) - \operatorname{val}\xi(\mathbf{n}')| = \operatorname{val}\widehat{F_i'}(\mathbf{n})$ for some $i \in \{1, 2, \ldots, d\}$. From the definition of F_i' it follows that $\operatorname{val}\widehat{F_i'}(\mathbf{n}) = \sum_{j=1}^p s_j \operatorname{val}\wp(\mathbf{a}^{(j)} \cdot (\mathbf{n}+\mathbf{x}) + \beta_j;\, a_i^{(j)})$. Note that $|\operatorname{val}\wp(\mathbf{a}^{(j)} \cdot (\mathbf{n}+\mathbf{x}) + \beta_j;\, a_i^{(j)})| \leq 1$, hence $|\operatorname{val}\xi(\mathbf{n}) - \operatorname{val}\xi(\mathbf{n}')| \leq \sum_{j=1}^p |s_j|$. $\square$

In order to show that $\operatorname{val}\varphi(\mathbf{n})$ is bounded, we prove three lemmas.

Lemma 2 *Assume that neither of* $\mathbf{n}', \mathbf{n}'' \in \mathbb{Z}^d$ *is special. Then there exists a path between them which contains no more than* $(2d-1)N$ *special points.*

Proof: By induction on d. If $d = 1$, there are N special points in all, so the claim is true. Assume that $d > 1$ and $\mathbf{n}' = (n_1', n_2', \ldots, n_d')$, $\mathbf{n}'' = (n_1'', n_2'', \ldots, n_d'')$. Consider the two discrete lines

$$L' = \{(n_1', n_2', \ldots, n_{d-1}', t)\,;\ t \in \mathbb{Z}\}, \quad L'' = \{(n_1'', n_2'', \ldots, n_{d-1}'', t)\,;\ t \in \mathbb{Z}\}.$$

Since $\mathbf{n}', \mathbf{n}''$ are not special, each of the lines L', L'' contains a finite number of special points, and there exists $t_0 \in \mathbb{Z}$ such that both $\mathbf{n}_0' = (n_1', n_2', \ldots, n_{d-1}', t_0)$ and $\mathbf{n}_0'' = (n_1'', n_2'', \ldots, n_{d-1}'', t_0)$ are not special. The straight path from $\mathbf{n}'$ to $\mathbf{n}_0'$ contains no more than N special points, as well as the straight path from $\mathbf{n}_0''$ to $\mathbf{n}''$. By induction hypothesis, there is a path in the set $\{(n_1, n_2, \ldots, n_{d-1}, t_0)\,;\ (n_1, n_2, \ldots, n_{d-1}) \in \mathbb{Z}^{d-1}\}$ from $\mathbf{n}_0'$ to $\mathbf{n}_0''$ that contains no more than $(2d-3)N$ special points. So there is a path from $\mathbf{n}'$ to $\mathbf{n}''$ that contains no more than $2N + (2d-3)N = (2d-1)N$ special points. $\square$

Lemma 3 *Let* $\mathbf{a} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}$, $\beta \in \mathbb{C}$, $\mathbf{q} \in \mathbb{Z}^d$, *and* $r \in \mathbb{N}_0$. *Denote*

$$A = \{\mathbf{n} \in \mathbb{Z}^d\,;\ \mathbf{a} \cdot \mathbf{n} = \beta, |n_i - q_i| \le r \text{ for } i = 1, 2, \ldots, d\}.$$

Then $|A| \le (2r+1)^{d-1}$.

Proof: Since $\mathbf{a} \neq \mathbf{0}$, there exists $k \in \{1, 2, \ldots, d\}$ such that $a_k \neq 0$. Denote $B = \{\mathbf{n} \in \mathbb{Z}^d;\ n_k = q_k, |n_i - q_i| \le r \text{ for } i = 1, 2, \ldots, d \text{ and } i \neq k\}$. The orthogonal projection $\pi : A \to B$, $\mathbf{n} \mapsto \mathbf{n} - (n_k - q_k)\mathbf{e}_k$ is injective, hence $|A| \le |B| = (2r+1)^{d-1}$. $\square$

Lemma 4 *Let* $\mathbf{n} = (n_1, n_2, \ldots, n_d) \in \mathbb{Z}^d$ *be special. Then there exists a non-special point* $\mathbf{n}^*$, *and a path from* $\mathbf{n}$ *to* $\mathbf{n}^*$ *which contains at most* $\frac{(N+1)d}{2}$ *special points.*

Proof: The set $\mathcal{P}$ of hyperplanes is finite, so not all points in $\mathbb{Z}^d$ are special. Let $r+1$ be the length of a shortest path from $\mathbf{n}$ to a non-special point $\mathbf{n}^*$. This path contains $r+2$ points, out of which at most $r+1$ are special. Notice that by the definition of r, the set

$$C_r = \{(\bar{n}_1, \bar{n}_2, \ldots, \bar{n}_d) \in \mathbb{Z}^d;\ |\bar{n}_i - n_i| \le r/d,\ i = 1, 2, \ldots, d\}$$

contains only special points. By Lemma 3, a hyperplane from $\mathcal{P}$ contains at most $(2\lfloor r/d \rfloor + 1)^{d-1}$ points from C_r, hence $|C_r| \le N(2\lfloor r/d \rfloor + 1)^{d-1}$. But $|C_r| = (2\lfloor r/d \rfloor + 1)^d$, so $N \ge 2\lfloor r/d \rfloor + 1 \ge 2(r-d+1)/d + 1$, and consequently $(N+1)d/2 \ge r+1$, which proves the assertion. $\square$

As a consequence we have

Proposition 7 *The sequence* $\operatorname{val}\xi(\mathbf{n})$ *is bounded on* $\mathbb{Z}^d$.

Proof: Let $\mathbf{n}_0 \in \mathbb{Z}^d$ be a fixed non-special point, and let $\mathbf{n} \in \mathbb{Z}^d$ be arbitrary. We distinguish two cases:

a) If $\mathbf{n}$ is non-special then, by Lemma 2, there exists a path from $\mathbf{n}_0$ to $\mathbf{n}$ that contains at most $(2d-1)N$ special points.

b) If $\mathbf{n}$ is special then, by Lemma 4, there is a non-special point $\mathbf{n}^*$ and a path from $\mathbf{n}$ to $\mathbf{n}^*$ that contains at most $(N+1)d/2$ special points. By Lemma 2, there is a path from $\mathbf{n}^*$ to $\mathbf{n}_0$ that contains at most $(2d-1)N$ special points.

In either case, there is a path from $\mathbf{n}_0$ to $\mathbf{n}$ that contains no more than $M := (2d-1)N + (N+1)d/2$ special points. By Proposition 6 we have then $|\operatorname{val}\xi(\mathbf{n}) - \operatorname{val}\xi(\mathbf{n}_0)| \leq M(|s_1| + |s_2| + \cdots + |s_p|)$. $\square$

Finally we get the following result.

Proposition 8 *The sequence* $\operatorname{val}\varphi(\mathbf{n})$ *is bounded on* $\mathbb{Z}^d$.

Proof: By Proposition 3(ii), $\operatorname{val}\varphi(\mathbf{n}) = \operatorname{val}\xi(\mathbf{n}) + \operatorname{val}\eta(\mathbf{n})$ for all $\mathbf{n} \in \mathbb{Z}^d$. So the sequence $\operatorname{val}\varphi(\mathbf{n})$ is bounded by Propositions 7 and 4. $\square$

Definition 7 *Let* $\mathcal{H}$ *be an H-system of the form (2).*

We say that a point $\mathbf{n}' \in \mathbb{Z}^d$ *is* accessible from a point $\mathbf{n} \in \mathbb{Z}^d$ w.r.t. $\mathcal{H}$ *if there exists a path* $\mathbf{n}_1, \mathbf{n}_2, \ldots, \mathbf{n}_k$ *such that* $\mathbf{n}_1 = \mathbf{n}$, $\mathbf{n}_k = \mathbf{n}'$ *and for each* $j \in \{1, 2, \ldots, k-1\}$ *there is* $i \in \{1, 2, \ldots, d\}$ *such that either* $\mathbf{n}_{j+1} = \mathbf{n}_j + \mathbf{e}_i$ *and* $f_i(\mathbf{n}_j) \neq 0$, *or* $\mathbf{n}_{j+1} = \mathbf{n}_j - \mathbf{e}_i$ *and* $g_i(\mathbf{n}_{j+1}) \neq 0$. *Otherwise* $\mathbf{n}'$ *is* inaccessible from $\mathbf{n}$ w.r.t. $\mathcal{H}$.

If $M \subseteq \mathbb{Z}^d$, *then* M *is* inaccessible w.r.t. $\mathcal{H}$ *if every* $\mathbf{n}' \in M$ *is inaccessible from any* $\mathbf{n} \in \mathbb{Z}^d \setminus M$ *w.r.t.* $\mathcal{H}$.

We will omit the qualification "w.r.t. $\mathcal{H}$" when the system $\mathcal{H}$ is clear from the context. Informally, $\mathbf{n}'$ is inaccessible from $\mathbf{n}$ if for any solution $T \in V_1(\mathcal{H})$, the value of T at $\mathbf{n}'$ is uniquely determined by $\mathcal{H}$ and the value of T at $\mathbf{n}$. Note also that the accessibility relation is reflexive and transitive.

Since the sequence $\operatorname{val}\varphi(\mathbf{n})$ is bounded on $\mathbb{Z}^d$, we can define

$$m = \min_{\mathbf{n} \in \mathbb{Z}^d} \operatorname{val}\varphi(\mathbf{n})$$

and associate with $\mathcal{H}$ the non-empty set

$$M_{\mathcal{H}} = \{\mathbf{n} \in \mathbb{Z}^d \mid \operatorname{val}\varphi(\mathbf{n}) = m\}.$$

Lemma 5 $M_{\mathcal{H}}$ *is inaccessible.*

Proof: It is sufficient to prove that if $\mathbf{a}$ is adjacent to $\mathbf{b}$, $\mathbf{a} \notin M_{\mathcal{H}}$ and $\mathbf{b} \in M_{\mathcal{H}}$, then $\mathbf{b}$ is inaccessible from $\mathbf{a}$. W.l.o.g. assume that $\mathbf{b} = \mathbf{a} + \mathbf{e}_1$. By (19) we have

$$\widehat{f_1}(\mathbf{a})\varphi(\mathbf{b}) = \widehat{g_1}(\mathbf{a})\varphi(\mathbf{a}). \tag{20}$$

By Proposition 3(ii), $\operatorname{val}\widehat{f_1}(\mathbf{a})+\operatorname{val}\varphi(\mathbf{b}) = \operatorname{val}\widehat{g_1}(\mathbf{a})+\operatorname{val}\varphi(\mathbf{a})$. As $\mathbf{a} \notin M_{\mathcal{H}}$ and $\mathbf{b} \in M_{\mathcal{H}}$, we have $\operatorname{val}\varphi(\mathbf{a}) > \operatorname{val}\varphi(\mathbf{b})$, therefore

$$\operatorname{val}\widehat{f_1}(\mathbf{a}) \;>\; \operatorname{val}\widehat{g_1}(\mathbf{a}).$$

Since $\operatorname{val}\widehat{g_1}(\mathbf{a}) \geq 0$, this implies that $\operatorname{val} f_1(\mathbf{a}+\mathbf{x}) = \operatorname{val}\widehat{f_1}(\mathbf{a}) > 0$. So by Proposition 3(i), $f_1(\mathbf{a}) = 0$. This proves that $\mathbf{b}$ is inaccessible from $\mathbf{a}$. $\square$

Lemma 6 *Let $\mathcal{H}$ be an H-system of the form (2). If $\mathbf{a}, \mathbf{b} \in M_{\mathcal{H}}$ are such that $\mathbf{b}$ is inaccessible from $\mathbf{a}$, then $\mathbf{a}$ is inaccessible from $\mathbf{b}$ as well.*

Proof: It suffices to prove the statement for the case where $\mathbf{a}$ is adjacent to $\mathbf{b}$. W.l.o.g. assume that $\mathbf{b} = \mathbf{a}+\mathbf{e}_1$. As in the proof of Lemma 5 we find that $\operatorname{val}\widehat{f_1}(\mathbf{a}) + \operatorname{val}\varphi(\mathbf{b}) = \operatorname{val}\widehat{g_1}(\mathbf{a}) + \operatorname{val}\varphi(\mathbf{a})$, but this time $\operatorname{val}\varphi(\mathbf{b}) = \operatorname{val}\varphi(\mathbf{a})$, so $\operatorname{val}\widehat{f_1}(\mathbf{a}) = \operatorname{val}\widehat{g_1}(\mathbf{a})$. Since $\mathbf{b}$ is inaccessible from $\mathbf{a}$, $f_1(\mathbf{a}) = 0$, which implies that $\operatorname{val}\widehat{f_1}(\mathbf{a}) > 0$. Hence $\operatorname{val}\widehat{g_1}(\mathbf{a}) > 0$ as well. Therefore $g_1(\mathbf{a}) = 0$, and the claim follows. $\square$

Theorem 5 *Let $\mathcal{H}$ be a consistent H-system. Then $\dim V_1(\mathcal{H}) > 0$.*

Proof: Pick any $\mathbf{a} \in M_{\mathcal{H}}$ and let $S(\mathbf{a}) = \{\mathbf{p} \in M_{\mathcal{H}}\,;\ \mathbf{p} \text{ is accessible from } \mathbf{a}\}$. We claim that $S(\mathbf{a})$ is inaccessible. Indeed, take $\mathbf{p} \in S(\mathbf{a})$ and $\mathbf{q} \notin S(\mathbf{a})$. Then either $\mathbf{q} \in M_{\mathcal{H}} \setminus S(\mathbf{a})$ or $\mathbf{q} \notin M_{\mathcal{H}}$. In the former case, $\mathbf{p}$ is inaccessible from $\mathbf{q}$ because otherwise, by Lemma 6, $\mathbf{q}$ is accessible from $\mathbf{p}$ and hence from $\mathbf{a}$, which is impossible since $\mathbf{q} \notin S(\mathbf{a})$. In the latter case, $\mathbf{p}$ is inaccessible from $\mathbf{q}$ because $\mathbf{p} \in M_{\mathcal{H}}$, $\mathbf{q} \notin M_{\mathcal{H}}$, and $M_{\mathcal{H}}$ is inaccessible by Lemma 5. This proves the claim.

Now define $T : \mathbb{Z}^d \to \mathbb{C}$ as follows. Set $T(\mathbf{a}) = 1$ and define T on $S(\mathbf{a}) \setminus \{\mathbf{a}\}$ recursively, using the system $\mathcal{H}$. This is possible because if $\mathbf{p} \in M_{\mathcal{H}}$ is accessible from $\mathbf{a}$ along some path $\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_k$ where $\mathbf{t}_1 = \mathbf{a}$ and $\mathbf{t}_k = \mathbf{p}$, then the entire path belongs to $M_{\mathcal{H}}$ (otherwise there is a j, $1 \leq j \leq k-1$, such that $\mathbf{t}_j \notin M_{\mathcal{H}}$, $\mathbf{t}_{j+1} \in M_{\mathcal{H}}$, and $\mathbf{t}_{j+1}$ is accessible from $\mathbf{t}_j$, which contradicts Lemma 5). Finally, set $T(\mathbf{p}) = 0$ for all $\mathbf{p} \notin S(\mathbf{a})$.

We claim that this T satisfies (2) for all $\mathbf{n} \in \mathbb{Z}^d$ and all $i \in \{1, 2, \ldots, d\}$. Indeed, if $\mathbf{n}, \mathbf{n}+\mathbf{e}_i \in S(\mathbf{a})$ then (2) is satisfied by definition of T and by consistency of $\mathcal{H}$. If $\mathbf{n}, \mathbf{n}+\mathbf{e}_i \notin S(\mathbf{a})$ then both sides of (2) are zero by definition of T. If $\mathbf{n} \in S(\mathbf{a})$ and $\mathbf{n}+\mathbf{e}_i \notin S(\mathbf{a})$ (or vice versa) then again both sides of (2) are zero by definition of T and because $S(\mathbf{a})$ is inaccessible. Hence $T \in V_1(\mathcal{H})$. Since $T \neq 0$, the claim follows. $\square$

Example 8 Let $\mathcal{H}$ be the system

$$\begin{aligned}(n_1+n_2+2)T(n_1+1, n_2) &= (n_1+n_2)(n_1-n_2)T(n_1, n_2),\\ (n_1+n_2+2)(n_1-n_2-1)T(n_1, n_2+1) &= (n_1+n_2)T(n_1, n_2).\end{aligned}$$

It is easy to check that $\mathcal{H}$ is a consistent H-system with certificates

$$F_1(n_1,n_2) = \frac{(n_1+n_2)(n_1-n_2)}{n_1+n_2+2} = \wp(n_1-n_2;\,1)\frac{R(n_1+1,n_2)}{R(n_1,n_2)},$$

$$F_2(n_1,n_2) = \frac{n_1+n_2}{(n_1+n_2+2)(n_1-n_2-1)} = \wp(n_1-n_2;\,-1)\frac{R(n_1,n_2+1)}{R(n_1,n_2)}$$

(cf. (13)), where

$$R(n_1,n_2) = \frac{1}{(n_1+n_2)(n_1+n_2+1)}.$$

Note that for $(n_1+n_2)(n_1+n_2+1)(n_1+n_2+2) \neq 0$, $\mathcal{H}$ is satisfied by

$$T(n_1,n_2) = \frac{(-1)^{n_1+n_2}}{\Gamma(1-n_1+n_2)}\,R(n_1,n_2),$$

but this solution does not belong to $V_1(\mathcal{H})$.

In this case $\xi(n_1,n_2)$ satisfies

$$\begin{aligned}
\xi(0,0) &= 1,\\
\xi(n_1+1,n_2) &= (n_1-n_2+x_1-x_2)\xi(n_1,n_2),\\
\xi(n_1,n_2+1) &= \frac{\xi(n_1,n_2)}{n_1-n_2-1+x_1-x_2}.
\end{aligned}$$

It is straightforward to verify that $\xi(n_1,n_2) = \wp(x_1-x_2;\,n_1-n_2)$ and

$$\operatorname{val}\xi(n_1,n_2) = \begin{cases} 0, & n_1 \le n_2,\\ 1, & \text{otherwise.}\end{cases}$$

Next, $\eta(n_1,n_2) = 1/((n_1+n_2+x_1+x_2)(n_1+n_2+1+x_1+x_2))$, and

$$\operatorname{val}\eta(n_1,n_2) = \begin{cases} -1, & (n_1+n_2)(n_1+n_2+1)=0,\\ 0, & \text{otherwise.}\end{cases}$$

Hence $m = \min_{(n_1,n_2)\in\mathbb{Z}^2} \operatorname{val}\varphi(n_1,n_2) = -1$, and

$$M_{\mathcal{H}} = \{(n_1,n_2)\in\mathbb{Z}^2\,;\ (n_1+n_2)(n_1+n_2+1)=0\ \wedge\ n_1\le n_2\}.$$

By taking $\mathbf{a} = (0,0)$ in the proof of Theorem 5, we have $S(\mathbf{a}) = M_{\mathcal{H}}$, and the corresponding non-zero solution belonging to $V_1(\mathcal{H})$ is

$$T(n_1,n_2) = \begin{cases} \frac{1}{\Gamma(2n_2+1)}, & n_1+n_2=0\ \wedge\ n_1\le n_2,\\ \frac{1}{\Gamma(2n_2+2)}, & n_1+n_2+1=0\ \wedge\ n_1\le n_2,\\ 0, & \text{otherwise.}\end{cases}$$

∎

6 The Ore-Sato theorem and its consequences

The Ore-Sato theorem (see Theorem 4) is commonly believed to imply that every hypergeometric term is of the form (1). For example, in [3, p. 223] one reads: "*From Ore's result it can be deduced that the most general form of A_{mn} is of the form*

$$A_{mn} = R(m,n)\gamma_{mn}a^m b^n$$

where R is a fixed rational function of m and n, a and b are constants, and γ_{mn} is a gamma product $(\dots)$ that is to say it is of the form

$$\gamma_{mn} = \prod_i \Gamma(a_i + u_i m + v_i n)/\Gamma(a_i)$$

where the a_i are arbitrary (real or complex) constants, and the u_i and v_i are arbitrary integers which may be positive, negative, or zero." A similar quote can be found in [4, p. 5].

It may be the case that in the literature referred to above, A_{mn} is implicitly assumed to be non-zero for all $m, n \in \mathbb{Z}$. This possibility is supported by the fact that, e.g., in [3] the corresponding H-system is given in terms of the two quotients $A_{m+1,n}/A_{mn}$ and $A_{m,n+1}/A_{mn}$. But such a severe restriction would exclude from consideration many important functions, such as the binomial coefficient $A_{mn} = \binom{m}{n}$, and all polynomials with integer roots.

However if we do not adopt this restriction, then there are hypergeometric terms which cannot be written in the form (1), as illustrated by the following example.

Example 9 Take the H-system

$$\begin{aligned} p(n_1,n_2)T(n_1+1,n_2) &= p(n_1+1,n_2)T(n_1,n_2), \qquad (21)\\ p(n_1,n_2)T(n_1,n_2+1) &= p(n_1,n_2+1)T(n_1,n_2), \end{aligned}$$

where $p(n_1,n_2) = (n_1 - n_2 - 1)(n_1 - n_2 + 1)$. It can be checked that any $T : \mathbb{Z}^2 \to \mathbb{C}$ which satisfies $T(n_1,n_2) = 0$ unless $n_1 = n_2$ is a solution of (21). In particular,

$$T(n_1,n_2) = \begin{cases} 2^{n_1^2}, & n_1 = n_2, \\ 0, & \text{otherwise} \end{cases}$$

is a solution of (21), even though it does not have the form (1) because it grows too fast along the diagonal. ■

There are examples which look less artificial and where the solution has Zariski-dense support, such as $T(n_1,n_2) = |n_1 - n_2|$. In [1, Example 6] it is shown that this hypergeometric term cannot be written in the form (1) if R is assumed to be a polynomial. In a similar way it can be shown that the same is true even if R is allowed to be a rational function.

The following statement does follow from the Ore-Sato theorem.

Corollary 2 *Let T be a hypergeometric term. If T has Zariski-dense support, then any constituent*[4] *of T is of the form (1).*

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[4] see Definition 5