Linear q-difference equations depending on a parameter

S. A. Abramov A. A. Ryabenko

Computing Centre of the Russian Academy of Sciences,
Vavilova 40, Moscow, 119333, Russia

Abstract

We consider linear q-difference equations with polynomial coefficients depending on a parameter. We discuss an algorithm recognizing the existence of numerical values of the parameter for which a given equation has a non-zero rational function solution. If such values exist, then the algorithm finds them as well as the corresponding solutions. In addition we propose parametric versions of the q-accurate summation, and q-Zeiberger algorithms. An implementation in Maple of all proposed algorithms is described.

Key words: q-difference equations with parameters, polynomial solutions, rational-function solutions, q-summation

1. Introduction

Suppose that in an equation $L(y) = 0$ the operator L is of the form

$$r_\rho(x, t_1, \dots, t_m)D^\rho + r_{\rho-1}(x, t_1, \dots, t_m)D^{\rho-1} + \cdots + r_0(x, t_1, \dots, t_m), \tag{1}$$

where $D = \frac{d}{dx}$, and $r_0, r_1, \dots, r_\rho$ are polynomials over $\mathbb{Q}$ in the specified variables, and $t_1, t_2, \dots, t_m$ are parameters. In the paper (Boucher, 1999) the following result of J.-A. Weil is mentioned: there is no algorithm that, for an arbitrary operator L of form (1) answers whether or not numerical values of parameters $t_1, t_2, \dots, t_m$ exist for which equation $L(y) = 0$ has a solution in the form of a non-zero rational function of x. The proof is based on the Davis-Matiyasevich-Putnam-Robinson theorem (Matiyasevich, 1993). The result by Weil can be easily extended to the problem of existence of polynomial solutions of equation $L(y) = 0$.

* This research was partly supported by the Russian Foundation for Basic Research, project no. 10-01-00249.

Email addresses: sergeyabramov@mail.ru (S. A. Abramov), ryabenko@cs.msu.ru (A. A. Ryabenko).

Similar results have been obtained for the difference case (Abramov (2009, 2010)). The operator L is of the form

$$r_\rho(x,t_1,\dots,t_m)E^\rho + r_{\rho-1}(x,t_1,\dots,t_m)E^{\rho-1} + \dots\dots + r_0(x,t_1,\dots,t_m), \qquad (2)$$

where E is the shift operator: $E(y(x)) = y(x+1)$, and again $r_0, r_1, \dots, r_\rho$ are polynomials over $\mathbb{Q}$ in the specified variables, $t_1, t_2, \dots, t_m$ are parameters.

In (Abramov, ISSAC'2010) q-difference equations with parameters were considered. Differential equations are based on the differentiation operator D, while difference equations are based on the shift operator E. In turn, the q-difference equations are based on the q-shift operator Q:

$$Q(y(x)) = y(qx),$$

where q is a fixed value or an additional variable (q-calculus and the theory and algorithms for q-difference equations are of interest in combinatorics, especially in the theory of partitions (Andrews, 1976, Sect. 8.4), (Andrews, 1986)). The q-difference analogue of operators (1), (2) is

$$r_\rho(x,t_1,\dots,t_m)Q^\rho + r_{\rho-1}(x,t_1,\dots,t_m)Q^{\rho-1} + \dots + r_0(x,t_1,\dots,t_m), \qquad (3)$$

where $r_0, r_1, \dots, r_\rho$ are polynomials in specified variables over a field k of characteristic 0. It is assumed that $k = k_0(q)$, where k_0 is a subfield of k, and q, x are algebraically independent over k_0. It was shown that in some sense the situation with the parametrized case for q-difference equations is more interesting than for differential and difference equations. Let, e.g., the ground field k be $\mathbb{Q}(q)$. Then there is an algorithm that recognizes the existence of numerical (real, complex) values of the parameters for which a given linear q-difference equation has a solution in the form of a non-zero polynomial or, alternatively, rational function; it is possible that the right-hand side is a non-zero polynomial in x that contains parameters (Abramov, ISSAC'2010, Sect. 4). At the same time, if the values of parameters are allowed to be arbitrary polynomials or rational functions of q then such algorithm does not exist (Abramov, ISSAC'2010, Sect. 5). The proof is based on two J. Denef's theorems (Denef, 1978).

Concerning the case of numeric values of parameters, it was emphasized in (Abramov, ISSAC'2010) that the aim of that paper was only to establish decidability of some algorithmic problems "in principle". In the current paper we restrict our consideration to the case of a single parameter, and propose for this case a detailed version of the algorithm from (Abramov, ISSAC'2010) that not only recognizes the existence of numerical values of the parameter for which a given equation has non-zero rational solutions, but also finds such values as well as the corresponding solutions (Sections 3.1, 3.2). This version of the algorithm uses only quite elementary algebraic tools like solving linear algebraic systems depending on a parameter. On the theoretical side, this is one of new results (in comparison with Abramov (ISSAC'2010)) presented in this paper. Another new theoretical result is a parameterized version of q-Zeilberger algorithm (Section 3.4).

A practical contribution of this paper is the Maple package `PQDEquations` whose main procedures are RationalSolution, QZeilberger, and AccurateQSummation (a parameterized version of the q-accurate summation is described in Section 3.3).

We touch upon the problem of finding corresponding values of the parameter in the fields of rational and algebraic functions of q. It was mentioned above that for the case of an arbitrary number of parameters the question related to rational functions is undecidable. For the case of a single parameter the question is open. In Section 4 we propose an approach that occasionally enables one to find adequate rational and algebraic function values of the parameter (the search for such values was not considered in Abramov (ISSAC'2010)). This is also implemented in the package `PQDEquations`, that contains some tools for experiments concerning adequate parameter values of functional type (Section 5.3).

2. Preliminaries

Hereafter we will suppose that $k = \mathbb{Q}(q)$, and q, x are algebraically independent over $\bar{\mathbb{Q}}$ (the algebraic numbers field). Let L be as in (3), and $f \in k[x, t_1, \dots, t_m]$. Considering the equation $L(y) = f$ we will suppose that $r_0, r_1, \dots, r_\rho$ are polynomials in $x, t_1, \dots, t_m$ over $\mathbb{Q}[q]$, and therefore the value $\deg_q r_i$ is defined for any $i = 0, 1, \dots, \rho$.

Let

$$w_q = \max_{i=0}^{\rho} \deg_q r_i, \;\; w_x = \max_{i=0}^{\rho} \deg_x r_i, \;\; w = \max\{w_q,\ w_x\}, \;\; d = \rho w_x^2 + 2 w_x w_q. \qquad (4)$$

The following two propositions are a consequence of (Abramov, ISSAC'2010, Prop. 2).

Proposition 1. Let the equation $L(y) = f$ have a polynomial solution $\varphi \in \bar{\mathbb{Q}}(q)[x]$ for some values $\tau_1, \tau_2, \dots, \tau_m \in \bar{\mathbb{Q}}$ of parameters $t_1,\ t_2,\ \dots\ , t_m$. Then

$$\deg_x \varphi \le \max\{w_q,\ \deg_x f\}. \qquad (5)$$

Proposition 2. Let the equation $L(y) = f$ for some values $\tau_1, \tau_2, \dots, \tau_m \in \bar{\mathbb{Q}}$ of parameters $t_1, t_2, \dots, t_m$ have a rational-function solution u. Let $\tau_1,\ \tau_2,\ \dots, \tau_m$ be such that $r_\rho(q, x, \tau_1, \tau_2, \dots, \tau_m)$ is a non-zero polynomial. Then u can be represented as $\frac{\varphi}{W(q,x,\tau_1,\tau_2,\dots,\tau_m)}$, where $\varphi \in \bar{\mathbb{Q}}(q)[x]$ and

$$W(q, x, \tau_1, \tau_2, \dots, \tau_m) = x^w \prod_{i=0}^{d} r_\rho(q, q^{-\rho-i}x, \tau_1, \tau_2, \dots, \tau_m).$$

Thus the value

$$l = \max\{w_q,\ \deg_x f\}. \qquad (6)$$

is an upper bound for the degree of polynomial solutions (the bound is independent of the values of parameters). If r_ρ does not vanish for concrete values of parameters then the polynomial

$$W(q, x, t_1, t_2, \dots, t_m) = x^w \prod_{i=0}^{d} r_\rho(q, q^{-\rho-i}x, t_1, t_2, \dots, t_m) \qquad (7)$$

is a universal denominator of all rational solutions of $L(y) = f$ for these values of parameters.

This plays a significant role in the sequel.

3. The case of a single parameter: finding rational solutions and related summation problems

Hereafter we will suppose that there is only one parameter, denoted by t. The corresponding q-difference operator is

$$r_\rho(x, t)Q^\rho + r_{\rho-1}(x, t)Q^{\rho-1} + \dots \cdots + r_0(x, t), \qquad (8)$$

$r_0, r_1, \ldots, r_\rho$ are polynomials over $\mathbb{Q}[q]$ in the specified variables, and

$$\gcd(r_\rho(x,t), \ldots, r_0(x,t)) = 1.$$

3.1. Polynomial solutions

Let L have the form (8), $f \in k[x,t]$, and l be as in (6). The method of undetermined coefficients can be used. Let $y_0, y_1, \ldots, y_l$ be the undetermined coefficients. We get a system S of linear algebraic equations for $y_0, y_1, \ldots, y_l$ with coefficients from $\mathbb{Q}[q,t]$, and try to find such value of t belonging to $\bar{\mathbb{Q}}$ that the system which is obtained as a result of substituting this values into S, has a non-zero solution with components in $\bar{\mathbb{Q}}(q)$. Working over the field $k(t)$ we use the Gaussian elimination for reducing S to a system S' whose matrix is in the row echelon form. Solve the system S separately for each of values of t such that at least one of the pivots vanishes when we substitute this value into it. Similarly consider separately the values of t such that all the right hand sides of the equations of S' having the form $0 = g$ vanish. All other values of the parameter can be considered as in the generic case.

Example 1. Consider the equation

$$y(q^2x)-(2t+q^2-x)y(qx)+(t+q^2)(-x+t)y(x) = -q(q-1)x^2-q(q^2+t-qt)x+q^2(-1+t).$$

The upper bound (6) is equal to 2. The system S is

$$\begin{aligned}
&(-1+t)(-1+t+q^2)\,y_0 && = (-1+t)q^2\\
&(-t-q^2+1)\,y_0 - (t-q+q^2)(-t+q)\,y_1 && = -q(q^2+t-qt)\\
&(q-t-q^2)\,y_1 - t(q^2-t)\,y_2 && = q(1-q)\\
&t\,y_2 && = 0
\end{aligned}$$

After the Gaussian elimination we get S'

$$\begin{aligned}
&(-1+t)(-1+t+q^2)\,y_0 && = (-1+t)q^2\\
&-(t-q+q^2)(-t+q)\,y_1 && = -q(q-1)(-t+q)\\
&-t(q^2-t)\,y_2 && = 0
\end{aligned}$$

Considering separately the values $t = 0, 1$ we find polynomial solutions

$$y_2x^2 + x + \frac{q^2}{q^2-1},$$

(y_2 is an arbitrary constant) for $t = 0$ and, resp.

$$1 + \frac{q(q-1)x}{q^2-q+1} \tag{9}$$

for $t = 1$. For all other algebraic values of t we get the solution

$$\frac{q^2}{q^2+t-1} + \frac{xq(q-1)}{q^2+t-q}. \tag{10}$$

Note that the solution (9) that we get for $t = 1$ is a particular case of (10).

3.2. *Rational solutions*

In the single parameter case the universal denominator (7) can be rewritten as

$$W(q,x,t) = x^w \prod_{i=0}^{d} r_\rho(q, q^{-\rho-i}x, t). \tag{11}$$

(subresultant methods from (Abramov & Kvashenko, 1993) can be used for decreasing the degree of (11)). The substitution $y(x) = \frac{z(x)}{W(q,x,t)}$ into the original equation reduces the problem of finding rational solutions to the problem of finding polynomial solutions. However each of values of t such that r_ρ vanishes has to be tested separately (in accordance to Proposition 2).

Example 2. By (11), for the equation

$$(xq + q^2 + t)y(qx) + (-x - q^2 - t)y(x) = 0$$

we get the universal denominator

$$W = x^2 \left(x + q^2 + t\right) \left(\frac{x}{q} + q^2 + t\right) \left(\frac{x}{q^2} + q^2 + t\right) \cdots \left(\frac{x}{q^5} + q^2 + t\right). \tag{12}$$

After substituting $y(x) = z(x)/W(q,x,t)$ into the given equation and clearing denominators we get the equation

$$(x + q^7 + tq^5)z(xq) + (-q^7t - q^7x - q^9)z(x) = 0.$$

having the polynomial solution

$$z(x) = C\, x^2(x + q^3 + t\, q)(x + q^4 + t\, q^2) \cdots (q^7 + x + t\, q^5)$$

for an arbitrary algebraic value of t (C is an arbitrary constant). The rational solution is $y(x) = \frac{C}{x+q^2+t}$.

Example 3. Consider the inhomogeneous equation with the operator L from the previous example:

$$(xq + q^2 + t)y(qx) + (-x - q^2 - t)y(x) = 1 - t^2.$$

We may use the universal denominator (12). After substituting $y(x) = z(x)/W(q,x,t)$ we get the equation

$$-q^8(x+q^7+tq^5)z(qx)+q^{15}(x+q^2+t)z(x) = x^2(t^2-1)(x+q^2+t)(x+q^3+qt)\cdots(x+q^7+tq^5).$$

having a polynomial solution iff $t = \pm 1$. The rational solution of the original equation is $\frac{C}{x+q^2+1}$ for $t = 1$, and $\frac{C}{x+q^2-1}$ for $t = -1$. This can be rewritten as $\frac{C}{x+q^2+t}$ for the case $t^2 - 1 = 0$.

In the following example non-rational algebraic numbers appear.

Example 4. With the same operator L, the inhomogeneous equation

$$(xq + q^2 + t)y(qx) + (-x - q^2 - t)y(x) = t^3 + t + 1$$

has a rational solution iff $t^3 + t + 1 = 0$: we have

$$\frac{C}{x + q^2 + t}$$

for the case $t^3 + t + 1 = 0$.

For the equations considered above there is no value of t such that the leading coefficient vanishes. In the following example there is one.

Example 5. For

$$(t+2)(xq + q^2 + t)y(qx) + (-x - q^2 - t)y(x) = t^2 - 1$$

there is $t = -2$ for which the given equation turns to

$$(-x - q^2 + 2)y(x) = 3$$

with the solution $\frac{3}{-x-q^2+2}$. For the case $t \neq 2$ the universal denominator is (12) and there are the solutions $\frac{C}{x+q^2-1}$ for $t = -1$ and $\frac{-1+t}{x+q^2+t}$ for any other values of t.

3.3. q-Accurate summation algorithm

Suppose that an operator L contains no parameters, i.e., $L \in k[x, Q]$. Then an operator $R \in k(x)[Q]$ is said to be a *summing operator* for L if

$$(Q - 1) \circ R = 1 + M \circ L$$

for some $M \in k(x)[Q]$. In this sense $R \equiv (Q-1)^{-1} \pmod{L}$.

We can assume w.l.g. that $\mathrm{ord}R < \mathrm{ord}L = \rho$. In the case $\rho = 1$ we have $\mathrm{ord}R = 0$, i.e., R is a rational function. If a summing operator exists then it can be constructed by the q-accurate summation algorithm (Abramov & van Hoei, 1997, 1999) or, when $\rho = 1$, by q-Gosper's algorithm (Petkovšek & Wilf & Zeilberger, 1996) (the q-accurate summation algorithm is therefore more general than q-Gosper's algorithm, and we will not consider q-Gosper's algorithm in the subsequent text). The summing operator R exists iff the equation

$$L^*(y) = 1 \tag{13}$$

has a rational solution $r \in k(x)$. The operator L^* in (13) is the adjoint operator for L. For

$$L = r_\rho Q^\rho + \cdots + r_1 Q + r_0,$$

one has

$$L^* = Q^{-\rho}(r_\rho)Q^{-\rho} + \cdots + Q^{-1}(r_1)Q^{-1} + r_0.$$

If a rational solution r exists then a summing operator R satisfies the relation

$$(Q-1)R = 1 - rL$$

($1 - rL$ is left divisible by $Q - 1$) and can be found easily.

This algorithm is used for finding indefinite sums of sequences of the form $f(q^n)$ where $f(x)$ is a solution of $L(y) = 0$ (the sequence $g(q^n)$ is an indefinite sum of $f(q^n)$ iff $g(q^{n+1}) - g(q^n) = f(q^n)$).

The algorithm from Section 3.2 enables us to use the q-accurate summation in the parameterized case.

Example 6. Consider the q-hypergeometric parameterized sequence

$$h_n = (tq)^n (\frac{q}{t}; q)_n, \tag{14}$$

where $(a;q)_n$ is the q-Pochhammer symbol:

$$(a;q)_n = \begin{cases} 1, & n = 0 \\ (1-a)(1-aq)\dots(1-aq^{n-1}), & n > 0 \end{cases}$$

Write x for q^n. Then

$$L = Q - q(t - qx)$$

annihilates h_n.

To find a summing operator we have to find a rational solution of

$$y\left(q^{-1}x\right) + q(-t + qx)y(x) = 1, \tag{15}$$

or, the same, a rational solution of

$$q(-t + q^2x)y(qx) + y(x) = 1.$$

This equation has a rational solution iff $t = 1$. This solution is $\frac{1}{xq^2}$. For $t = 1$ we find the summing operator

$$R = -\frac{1}{qx} = -\frac{1}{qq^n},$$

and the indefinite sum $-\frac{(q;q)_{n+1}}{q}$.

3.4. q-Zeilberger algorithm

For a function F of two variables x_1, x_2 we define

$$Q_1(F(x_1, x_2)) = F(qx_1, x_2),\ Q_2(F(x_1, x_2)) = F(x_1, qx_2).$$

Let H_1, H_2 be first order operators

$$H_1 = Q_1 - s_1(x_1, x_2),\ H_2 = Q_2 - s_2(x_1, x_2), \tag{16}$$

$s_1, s_2 \in k(x_1, x_2)$. A pair

$$(r(x_1, x_2), L),$$

$r(x_1, x_2) \in k(x_1, x_2)$, $L \in k(x_2)[Q_2]$, is a *Z-pair* of H_1, H_2, if

$$(Q_1 - 1)r = L + AH_1 + BH_2,$$

$A, B \in k(x_1, x_2)[Q_1, Q_2]$. The order of a Z-pair is defined as $\mathrm{ord}L$. For given $\rho \in \mathbb{N}$ and H_1, H_2 of the form (16) the q-Zeilberger algorithm (Petkovšek & Wilf & Zeilberger, 1996) recognizes the existence of a Z-pair of order ρ of H_1, H_2. If such a pair exists, then the algorithm finds it. The base of this algorithm is the fact that if L is of the form

$$L = r_\rho(x_2)Q_2^\rho + \dots + r_1(x_2)Q_2 + r_0(x_2)$$

then

$$s_1(x_1, x_2)r(qx_1) - r(x_1) = \sum_{i=0}^{\rho} r_i(x_2) \prod_{j=0}^{i-1} s_2(x_1, q^j x_2). \tag{17}$$

This equality is considered as a first order linear q-difference equation for $r(x_1)$, while the ground field is $k(x_2)$. One can construct a universal denominator (without applying Gosper's algorithm) which does not contain $r_0, r_1, \dots, r_\rho$. The corresponding substitution gives a first order equation for a polynomial numerator, and the usual approach gives an upper bound for the degree of this polynomial (the bound does not depend on

$r_0, r_1, \ldots, r_\rho$). The undeterminate coefficients method leads to a system of linear algebraic equations whose unknowns are $r_0, r_1, \ldots, r_\rho$ and the coefficients of a polynomial solution.

This algorithm is used for finding definite sums of q-hypergeometric two-dimensional sequences (terms). Recall that the sequence $g(q^n)$ is the definite sum of $f(q^m, q^n)$ iff $g(q^n) = \sum_{m=-\infty}^{\infty} f(q^m, q^n)$, or, alternatively $g(q^n) = \sum_{m=0}^{n} f(q^m, q^n)$.

The algorithm from Section 3.2 enables us to use the q-Zeilberger algorithm in the parameterized case.

Example 7. Consider the q-hypergeometric sequence

$$f(q^m, q^n) = (-1)^m q^{mn+\frac{3}{2}m+n-\frac{1}{2}m^2} \frac{(\frac{1}{q^n}; q)_m}{(q+qt; q)_m}.$$

Write x_1 for q^m and x_2 for q^n. Then

$$H_1 = Q_1 - \frac{q(x_2 - x_1)}{x_1(-1 + qx_1 + qx_1 t)},\ H_2 = Q_2 - \frac{x_1 q(qx_2 - 1)}{qx_2 - x_1}$$

annihilate $f(x_1, x_2)$. For $\rho = 0$ the equation (17) is

$$\frac{q(x_2 - x_1)}{x_1(-1 + x_1 + x_1 t)} r(qx_1) - r(x_1) = r_0.$$

There is no rational solution for any values of t with $r_0 \neq 0$. For $\rho = 1$ the equation (17) is

$$\frac{q(qx_2 - x_1)}{x_1(-1 + x_1 + x_1 t)} r(qx_1) - r(x_1) = r_0 + r_1 \frac{x_1 q(qx_2 - 1)}{qx_2 - x_1}.$$

It has a rational solution with $r_1 \neq 0$ iff $t = -1$. Finally, a first order Z-pair of H_1, H_2 is

$$\left(-\frac{(qx_2 - 1)x_1 q}{qx_2 - x_1},\ \ Q_2 + q^3 x_2 - q^2 \right).$$

Therefore if $t = -1$ then the sequence

$$g(q^n) = \sum_{m=0}^{n-1} f(q^m, q^n),$$

satisfies the equation

$$g(q^{n+1}) + (q^3 q^n - q^2) g(q^n) = q^{n+1}.$$

4. When t depends on q

We have mentioned in Section 1 that if the values of parameters are allowed to be arbitrary polynomials or rational functions of q and the number of parameters is arbitrary then there is no algorithm for recognizing the existence of such values of the parameters for which a given equation has a non-zero rational solution. We also mentioned that for the case of a single parameter the corresponding question is open. Below we propose an approach that occasionally enables one to find adequate rational and algebraic function values of the parameter t. Note that no upper bound for the degree of polynomial solutions, independent of functional values of t, exists in general. (For example the homogeneous q-difference equation $y(qx) - ty(x) = 0$ has the polynomial solution x^m of degree m when $t = q^m$, $m \in \mathbb{N}$.) However sometimes the use of the bound (5) yields results. Suppose that this bound is computed and we apply the algorithms from Sections 3.1, 3.2. The point is that for finding the values of t for which a polynomial $a \in \mathbb{Q}[q, t, x]$ vanishes we can represent a as a polynomial in x, and find the full factorization of the gcd of its coefficients. Let $p(q, t)$ be one of the irreducible factors. If $\deg_t p = 0$ then we ignore $p(q, t)$. Otherwise, if $\deg_q p = 0$ then this factor defines a value of t belonging to $\bar{\mathbb{Q}}$; if $\deg_q p > 0$ and $\deg_t p = 1$ then we get a value belonging to $\mathbb{Q}(q)$; if $\deg_q p > 0$ and $\deg_t p > 1$ then the obtained value belongs to $\overline{\mathbb{Q}(q)} \setminus \mathbb{Q}(q)$.

It is guaranteed that at least all the adequate numeric values will be found if we use the bounds given in Section 2. This is a motivation for the choice of these bounds in the functional case. Of course, any larger bounds also can be taken.

In the following examples we use the bounds from Section 2. We will return to this question in Section 5.3.

Example 8. For the equation from the Example 5 we get additionally $\frac{C}{x(x+q^2+q-2)} + \frac{q-3}{x+q^2+q-2}$ for $t = q-2$ and $\frac{C}{x^2(x+2q^2-2)} + \frac{q^2-3}{x+2q^2-2}$ for $t = q^2-2$. But the existing solutions for $t = q^3 - 2$, $t = q^4 - 2, \ldots$ are not obtained.

Example 9. For

$$y(qx) - t^2 q y(x) = 0$$

we get solutions $y(x) = C_1 x$ for $t = \pm 1$, C_0 if $t^2 q - 1 = 0$ and $\frac{C}{x}$ for $t = \pm \frac{1}{q}$. But the solution $\frac{C}{x^2}, \frac{C}{x^3}, \ldots$ for $t^2 q^2 - 1 = 0$, $t^2 q^3 - 1 = 0, \ldots$ are not obtained.

Example 10. Consider again the q-hypergeometric sequence (14). For the equation (15) we get $\frac{1}{q^2 x}$ for $t = 1$, $\frac{q+qx-1}{x^2 q^3}$ for $t = q$, and $\frac{qx^2-x+q^2x+1-q-q^2+q^3}{q^3x^3}$ for $t = q^2$. The bound (5) does not allow to get rational solutions for $t = q^3$, $t = q^4, \ldots$

Example 11. Consider the q-hypergeometric sequence

$$f(q^m, q^n) = (-1)^m q^{mn + \frac{3}{2}m + n - \frac{1}{2}m^2} \frac{(\frac{1}{q^n}; q)_m}{(q+t; q)_m}.$$

Write x_1 for q^m and x_2 for q^n. Then

$$H_1 = Q_1 - \frac{q(x_2 - x_1)}{x_1(-1 + qx_1 + x_1 t)}, \; H_2 = Q_2 - \frac{x_1 q(qx_2 - 1)}{qx_2 - x_1}$$

annihilate $f(x_1, x_2)$. For $\rho = 0$ there is no Z-pair. For $\rho = 1$ a Z-pair of H_1, H_2 is

$$\left(-\frac{(qx_2 - 1)x_1 q}{qx_2 - x_1}, \;\; Q_2 - q^3 x_2 + q^2 \right)$$

if $t = -q$.

5. PQDEquations package

We implemented the algorithms in the `PQDEquations` package. The package contains three procedures:

RationalSolution, AccurateQSummation, QZeilberger.

All these procedures use the corresponding procedures from the Maple package `QDifferenceEquations` (see (Maple, 2010)) if the input is not dependent on a parameter. The procedure GaussianElimination from the package `LinearAlgebra` is used as an auxiliary tool for finding polynomial solutions.

5.1. When t is a number

The output of the procedures may have one of the following three forms. The first output form is

$$t = t_1, F_1, \ \ldots, \ t = t_s, F_s$$

with $t_i \in \bar{\mathbb{Q}}$. Non-rational algebraic numbers are represented using Maple's RootOf mechanism. The form of F_i is dependent on the concrete procedure.

The second output form is

$$t = t_1, F_1, \ \ldots, \ t = t_s, F_s, \ p(t) \neq 0, F_{s+1},$$

where $p(t) \in \mathbb{Q}[t]$. The result is F_{s+1} for any $t \in \bar{\mathbb{Q}}$ which is not a root of $p(t)$. The polynomial $p(t)$ must have the roots t_i, $i = 1, 2, \ldots, s$ (some extra roots are also possible).

The third output form is F. This means that we have F for any $t \in \bar{\mathbb{Q}}$.

For the RationalSolution procedure the input equation must be given in the form which is recognizable by the procedure QECreate from the `QDifferenceEquations` package. We illustrate the use of RationalSolution by the equations from Examples 1, 2, 3, 4, and 5.

```
> with(PQDEquations):
> Exmpl_1 := y(q^2*x)-(2*t+q^2-x)*y(q*x)+(t+q^2)*(-x+t)*y(x) =
>            -q*(q-1)*x^2-q*(q^2+t-q*t)*x+q^2*(-1+t):
> RationalSolution(Exmpl_1, y(x), t);
```

$$t = 0, \_C_2\, x^2 + x + \frac{q^2}{q^2-1}, \ t = 1, 1 + \frac{(-1+q)qx}{1+q^2-q}, \ t(t-1) \neq 0, \frac{q^2}{q^2+t-1} + \frac{xq(q-1)}{q^2+t-q}$$

Names $\_C_i$ (where i is integer) denote arbitrary constants. The CPU time [1] needed to compute is 0.140.

```
> Exmpl_2 := (-x-q^2-t)*y(x)+(x*q+q^2+t)*y(x*q):
> RationalSolution(Exmpl_2, y(x), t);
```

$$\frac{q^{15}\_C_7}{x+q^2+t}$$

The CPU time is 0.056.

```
> RationalSolution(Exmpl_2 = 1 - t^2, y(x), t);
```

$$t = 1, \frac{q^{15}\_C_7}{x+q^2+1}, \ t = -1, \frac{q^{15}\_C_7}{x+q^2-1},$$

The CPU time is 0.132.

```
> RationalSolution(Exmpl_2 = t^3+t+1, y(x), t);
```

$$t = RootOf(1 + \_Z^3 + \_Z), \frac{q^{15}\_C_7}{RootOf(1 + \_Z^3 + \_Z) + x + q^2}$$

The CPU time is 0.068.

---

[1] For all the experiments: Maple 13, Ubuntu 8.04.4 LTS, AMD Athlon(tm) 64 Processor 3700+, 990MB.

```
> Exmpl_5 := (-x-q^2-t)*y(x)+(t+2)*(q*x+q^2+t)*y(q*x) = t^2-1:
> RationalSolution(Exmpl_5, y(x), t);
```

$$t=-2,-\frac{3}{x+q^2-2}, t=-1,-\frac{\_C_1}{x+q^2-1}, (t+2)(t+1)\neq 0, \frac{t-1}{x+q^2+t}$$

The CPU time is 0.176.

The first argument of the procedure AccurateQSummation is a q-difference operator which must be a polynomial in x and Q. In the output, F_i (as well as F_{s+1} and F) is a summing operator R. Return back to the Example 6:

```
> AccurateQSummation(Q-q*(t-q*x), Q, x, q, t);
```

$$t=1,-\frac{1}{qx}$$

The CPU time is 0.152.

The arguments of the QZeilberger procedure are rational functions s_1, s_2, and the order ρ of the constructed Z-pair. In the output, F_i is a list $[L, r]$, where L is a q-difference operator with q-shift operator Q_2 while r is a rational function.

```
> QZeilberger(q*(x[1]-x[2])/((-1+q*x[1]+q*x[1]*t)*x[1]),
>              -x[1]*q*(q*x[2]-1)/(q*x[2]-x[1]), x[1], x[2], Q[2], 0, t);
```

No Z-pair of order 0 was found

The CPU time is 0.228.

```
> QZeilberger(q*(x[1]-x[2])/((-1+q*x[1]+q*x[1]*t)*x[1]),
>              -x[1]*q*(q*x[2]-1)/(q*x[2]-x[1]), x[1], x[2], Q[2], 1, t);
```

$$t=-1,\left[q^3x_2-q^2+Q_2, \frac{x_1q(qx_2-1)}{qx_2-x_1}\right]$$

The CPU time is 0.236.

5.2. When t depends on q

The flag

```
> SetExtended(true):
```

initializes the search for adequate functional values of the parameter. If this flag is set then we get the following results from the examples considered above

```
> RationalSolution(Exmpl_5, y(x), t);
```

$$t=-2,-\frac{3}{x+q^2-2}, t=-1,-\frac{\_C_1}{x+q^2-1},$$
$$t=q-2, \frac{qx-3x-\_C_1}{x(x+q^2+q-2)}, t=q^2-2, \frac{x^2q^2-3x^2-\_C_1}{(x+2q^2-2)x^2},$$
$$(t+2)(1+t)(q-t-2)(q^2-2-t)\neq 0, \frac{t-1}{x+q^2+t}$$

The CPU time is 0.376.

```
> RationalSolution(y(q*x)-t^2*q*y(x), y(x), t);
```

$$
\begin{aligned}
&t = 1, \_C_1 x, t = -1, \_C_1 x,\\
&t = RootOf(-1 + \_Z^2 q), \_C_1,\\
&t = \frac{1}{q}, \frac{\_C_0}{x}, t = -\frac{1}{q}, \frac{\_C_0}{x}
\end{aligned}
$$

The CPU time is 0.092.

```
> AccurateQSummation(Q-q*(t-q*x), Q, x, q, t);
```

$$
t = 1, -\frac{1}{qx}, t = q, -\frac{q + x - 1}{qx^2}, t = q^2, -\frac{x^2 + q^4 - q^3 + q^2 x - q^2 + q - x}{qx^3}
$$

The CPU time is 0.128.

```
> QZeilberger(q*(x[1]-x[2])/((-1+q*x[1]+x[1]*t)*x[1]),
>            x[1]*q*(q*x[2]-1)/(q*x[2]-x[1]), x[1], x[2], Q[2], 1, t);
```

$$
t = -q, \left[-q^3 x_2 + q^2 + Q_2, -\frac{x_1 q (q x_2 - 1)}{q x_2 - x_1}\right]
$$

The CPU time is 0.284.

5.3. *Use of larger bounds*

Our package gives a possibility to increase the bounds l, w, and d. The user can indicate a non-negative integer N that will be added to the computed bounds. For example, due to the command

```
> SetIncrement(2):
```

the program will use

$$
w = \max\{w_q,\ w_x\} + 2,\quad d = \rho w_x^2 + 2 w_x w_q + 2,\quad l = \max\{w_q,\ \deg_x f\} + 2.
$$

Occasionally this enables one to find some additional adequate rational and algebraic function values of the parameter t. Going back to the first example from Section 5.2 we perform the computation based on the new bounds:

```
> RationalSolution(Exmpl_5, y(x), t);
```

$$
\begin{aligned}
&t = -2, -\frac{3}{x + q^2 - 2}, t = -1, -\frac{\_C_1}{x + q^2 - 1},\\
&t = q - 2, \frac{qx - 3x - \_C_1}{x(x + q^2 + q - 2)}, t = q^2 - 2, \frac{x^2 q^2 - 3x^2 - \_C_1}{(x + 2q^2 - 2)x^2},\\
&t = q^3 - 2, \frac{x^3 q^3 - 3x^3 - \_C_1}{(x + q^2 + q^3 - 2)x^3}, t = q^4 - 2, \frac{x^4 q^4 - 3x^4 - \_C_1}{(x + q^2 + q^4 - 2)x^4},\\
&t = -\frac{-1 + 2q^2}{q^2}, -\frac{\_C_1 x^2 q^2 + 3q^2 - 1}{q^2 x + q^4 + 1 - 2q^2}, t = -\frac{2q - 1}{q}, -\frac{\_C_1 x q + 3q - 1}{qx + q^3 - 2q + 1},\\
&(q^2 t - 1 + 2q^2)(qt - 1 + 2q)(t + 1)(t + 2)(-t - 2 + q)\,(-t - 2\\
&\quad + q^2)\,(-t - 2 + q^3)(-t - 2 + q^4) \neq 0, \frac{-1 + t}{x + q^2 + t}
\end{aligned}
$$

Now we know sufficiently many solutions for different values of the parameter to make the conjecture that for

$$t = q^n - 2, \quad \text{and} \quad t = \frac{1 - 2q^n}{q^n}$$

the equation has rational solutions

$$\frac{x^n q^n - 3x^n - \_C_1}{x^n(x + q^2 - 2 + q^n)} \quad \text{and} \quad - \frac{\_C_1 x^n q^n + 3q^n - 1}{q^n(x + q^2 - 2 + q^{-n})},$$

$n = 0, 1, 2 \ldots$ We can formulate this conjecture more succinctly: if $t = q^n - 2$ then the original equation has the rational solution

$$\frac{x^n q^n - 3x^n - \_C_1}{x^n(x + q^2 - 2 + q^n)}$$

for all integer n. The latter conjecture can be verified by Maple. We substitute the assumed solution into the equation by the standard procedure `eval`, and simplify the result by `simplify`. This yields the equality $0 = 0$:

```
> eval(Expml_5, {t = q^n-2,
>          y(x) = (q^n*x^n-3*x^n-_C[1])/(x^n*(x+q^2-2+q^n)),
>     eval(y(x) = (q^n*x^n-3*x^n-_C[1])/(x^n*(x+q^2-2+q^n)), x = q*x)});
> simplify((lhs-rhs)(%)=0) assuming n::integer;
```

$$\frac{(2 - x - q^2 - q^n)(q^n x^n - 3x^n - \_C_1)}{x^n(x + q^2 - 2 + q^n)} + \frac{q^n(q^n(qx)^n - 3(qx)^n - \_C_1)}{(qx)^n} = (q^n - 2)^2 - 1$$

$$0 = 0$$

The package `PQDEquations` is available from

http://www.ccas.ru/sabramov/PQDE

Acknowledgments

The authors are thankful to D. Khmelnov and M. Petkovšek for useful discussions, and to anonymous referees for helpful comments.

References

Abramov, S. A., 2009. On an undecidable problem connected with differential and difference equations. *Proceedings of XIII International conference on differential equations (Erugin readings - 2009)*, May 26-29, 2009, Pinsk, Belarus, p. 118 (in Russian).

Abramov, S. A., 2010. On an undecidable problem related to difference equations with parameters, *Programming and Computer Software*, **36**, No. 2.

Abramov, S. A., 2010. On some decidable and undecidable problems related to q-difference equations with parameters. *ISSAC'2010 Proceedings*, 311–317.

Abramov, S. A., van Hoeij, M., 1997. A method for the integration of solutions of Ore equations, *ISSAC'97 Proceedings*, 172–175.

Abramov, S. A., van Hoeij, M., 1999. Integration of solutions of linear functional equations, *Integral Transformations and Special Functions*, **8**, No 1-2, 3–12.

Abramov, S. A., Kvashenko, K. Yu., 1993. On the greatest common divisor of polynomials which depend on a parameter, *ISSAC'93 Proceedings*, 152–156.

Andrews, G. E., 1976. The Theory of Partitions. *Encyclopedia of Mathematics and its Applications*, Vol. 2, Addison-Wesley, Reading, Mass.

Andrews, G. E., 1986. q-Series: Their Development and Application in Analysis, Number Theory, Combinatorics, Physics, and Computer Algebra. *CBMS Regional Conference Series*, No. 66, AMS, R.I.

Boucher, D., 1999. About the polynomial solutions of homogeneous linear differential equations depending on parameters, *ISSAC'99 Proceedings*, 261–268.

Denef, J., 1978. The diophantine problem for polynomial rings and fields of rational functions, *Transactions of the American Mathematical Society*, **242**, 391–399.

Matiyasevich, Yu. V., 1993. Hilbert's Tenth Problem. *MIT Press*, Cambrige, MA.

Petkovšek, M., Wilf, H. S., and Zeilberger, D., 1996. $A = B$, *A K Peters*.

Maple online help: http://www.maplesoft.com/support/help/