On summation of P-recursive sequences*

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Abstract

We consider sequences which satisfy a linear recurrence equation $Ly = 0$ with polynomial coefficients. A criterion, i.e., a necessary and sufficient condition is proposed for validity of the discrete Newton-Leibniz formula when a primitive (an indefinite sum) Rt of a solution t of $Ly = 0$ is obtained either by Gosper's algorithm or by the Accurate Summation algorithm (the operator R has rational-function coefficients, $\operatorname{ord} R = \operatorname{ord} L - 1$; in the Gosper case $\operatorname{ord} L = 1$, $\operatorname{ord} R = 0$). Additionally we show that if Gosper's algorithm succeeds on L, $\operatorname{ord} L = 1$, then $Ly = 0$ *always* has some non-zero solutions t, defined everywhere, such that the discrete Newton-Leibniz formula $\sum_{k=v}^{w} t(k) = u(w+1) - u(v)$ is valid for $u = Rt$ and any integer bounds $v \leq w$.

1 Introduction

Let K be a field of characteristic zero. If $t(k)$ is a K-valued sequence, then $Et(k)$ is the sequence $s(k) = t(k+1)$. We consider *P-recursive* sequences, i.e., sequences, that satisfy recurrence equations of the form $Ly = 0$, where

$$L = a_\rho(k)E^\rho + a_{\rho-1}(k)E^{\rho-1} + \cdots + a_0(k), \tag{1}$$

$\rho \geq 1$, $a_\rho(k), a_{\rho-1}(k), \ldots, a_0(k) \in K[k]$, $a_\rho(k)a_0(k) \not\equiv 0$ and $\gcd(a_0(k), \ldots, a_{\rho-1}(k), a_\rho(k)) = 1$. If $\operatorname{ord} L = \rho = 1$, then the corresponding P-recursive sequences are *hypergeometric terms*.

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In [4] we discussed validity of the discrete Newton-Leibniz formula when an indefinite sum of the sequence $t(k)$ is obtained either by Gosper's algorithm [5] or by the Accurate Summation algorithm [3]. These algorithms, which we denote hereafter by $\mathcal{GA}$ and $\mathcal{AS}$, respectively, search for a solution u of the *telescoping equation*

$$Eu(k) - u(k) \;=\; t(k) \tag{2}$$

where the sequence $t(k)$ is P-recursive and satisfies $Lt = 0$. Suppose that using one of these algorithms we found a linear recurrence operator R of order $\operatorname{ord} L - 1$ with rational-function coefficients, such that $u = Rt$ is a solution of (2) for some solution t of $Lt = 0$ (in the Gosper case $\operatorname{ord} L = 1$ and $\operatorname{ord} R = 0$, i.e., R is a rational function). Then the question is: can we use the discrete Newton-Leibniz formula

$$\sum_{k=v}^{w} t(k) = u(w+1) - u(v) \tag{3}$$

to find the definite sum of values of t? It was shown in [4] that sometimes (3) is not valid even when all of $t(v), t(v+1), \ldots, t(w),\ u(v), u(w+1)$ are defined. The reason is that equation (2) may fail to hold at certain points k of the summation interval.

Both $\mathcal{GA}$, $\mathcal{AS}$ start by constructing the minimal annihilator L of a given concrete sequence t and this step is not formalized. On the next steps these algorithms work with L only, while the sequence t itself is ignored (more precisely, in the case of $\operatorname{ord} L = 1$, $L = a_1(k)E + a_0(k)$, $\mathcal{GA}$ works with the certificate of t, i.e., with the rational function $-\frac{a_0(k)}{a_1(k)}$). The algorithms try to construct an operator R such that $u = Rt$ in (2).

If L is of type (1) then denote by $V(L)$ the space of all sequences $t(k)$ defined for all $k \in \mathbb{Z}$ and such that $Lt = 0$. If additionally R is obtained from L either by $\mathcal{GA}$ or by $\mathcal{AS}$, then denote by $V_R(L)$ the subspace of $V(L)$ which contains $t \in V(L)$ iff formula (3) is valid for any integer $v \le w$ with $u = Rt$.

It may be that $\dim V_R(L) < \dim V(L)$ (notice that quite often $\dim V(L) > \operatorname{ord} L$; it is possible that $\dim V_R(L) > \operatorname{ord} L$ as well).

In [4] some sufficient conditions for validity of (3) for a given sequence were given. In this paper we present a criterion, i.e., a necessary and sufficient condition for validity of this formula for all $k \in \mathbb{Z}$ when $u = Rt$, and R is obtained either by $\mathcal{GA}$ or by $\mathcal{AS}$. Note that (3) is valid for all integer bounds $v \le w$ iff (2) is valid for all $k \in \mathbb{Z}$. In addition, if R is obtained either by $\mathcal{GA}$ or by $\mathcal{AS}$, then we present a description of the linear space $V_R(L)$, and prove that in the case of $\mathrm{ord} L = 1$ the dimension of $V_R(L)$ is always positive.

We assume that $K = \mathbb{C}$ in all examples of this paper.

Example 1 *$\mathcal{GA}$ succeeds on the operator $L = kE - (k+1)^2$, and the result is $R = \frac{1}{k}$. The space $V(L)$ is two-dimensional: the sequences*

$$t_1(k) = \begin{cases} 0, & \text{if } k < 0, \\ k \cdot k!, & \text{if } k \geq 0 \end{cases}$$

and

$$t_2(k) = \begin{cases} \frac{(-1)^k k}{(-k-1)!}, & \text{if } k < 0, \\ 0, & \text{if } k \geq 0 \end{cases}$$

form a basis of $V(L)$. Our criterion says that, generally speaking, (3) is not applicable to t_1, but is applicable to t_2. We can illustrate this as follows. Applying (3) to t_1 with $v = -1, w = 1$, we have

$$t_1(-1) + t_1(0) + t_1(1) = \frac{1}{k} t_1(k) \mid_{k=2} - \frac{1}{k} t_1(k) \mid_{k=-1} = \frac{1}{2} \cdot 4 - 0 = 2$$

which is wrong, because $t_1(-1) + t_1(0) + t_1(1) = 0 + 0 + 1 = 1$. Applying (3) to t_2 with the same v, w, we have

$$t_2(-1) + t_2(0) + t_2(1) = \frac{1}{k} t_2(k) \mid_{k=2} - \frac{1}{k} t_2(k) \mid_{k=-1} = 0 - (-1) = 1$$

which is correct, because $t_2(-1) + t_2(0) + t_2(1) = 1 + 0 + 0 = 1$.

Our algorithm computes a basis of the subspace $V_R(L)$. In Example 1 $V_R(L)$ is one-dimensional and is generated by t_2. Examples which demonstrate that sometimes this dimension can be greater than 1 are given (see Examples 2, 4).

2 Preliminaries

For $f(k), g(k) \in K[k]$ we write $f(k) \perp g(k)$ to indicate that $f(k)$ and $g(k)$ are coprime. If $r(k) \in K(k)$, then $\mathrm{den}(r(k))$ is the monic polynomial from $K[k]$ such that $r(k) = \frac{f(k)}{\mathrm{den}(r(k))}$ for some $f(k) \in K[k]$, $f(k) \perp \mathrm{den}(r(k))$. If L and M are linear recurrence operators with coefficients from $K(k)$ then we write

$L \circ M$ for the product of L and M in the non-commutative ring $K(k)[E]$. If $M = r(k)$ is a rational function, then $L \circ r(k)$ is an operator of the same order as L, while $Lr(k)$ is a rational function (the result of applying L to $r(k)$).

The algorithm $\mathcal{AS}$ starts with finding a rational function solution $r(k)$ of the equation $L^* y = 1$ (say, by the algorithms from [1] or [2]), where L^* is the adjoint of L:

$$L^* = a_\rho(k-\rho)E^{-\rho} + a_{\rho-1}(k-\rho+1)E^{-\rho+1} + \cdots + a_0(k).$$

The equation satisfied by the rational function $r(k)$ can be rewritten as

$$a_0(k+\rho)r(k+\rho) + a_1(k+\rho-1)r(k+\rho-1) + \cdots + a_\rho(k)r(k) = 1. \quad (4)$$

If such $r(k)$ exists then R can be found from the relation

$$1 - rL = (E-1) \circ R. \quad (5)$$

We obtain

$$R = c_{\rho-1}(k)E^{\rho-1} + c_{\rho-2}(k)E^{\rho-2} + \cdots + c_0(k), \quad (6)$$

where

$$c_i(k) = \sum_{j=0}^{i} r(k+j)a_{i-j}(k+j) - 1 \quad (7)$$

for $0 \le i \le \rho - 1$.

$\mathcal{GA}$ works with the case $\rho = 1$,

$$L = a_1(k)E + a_0(k), \;\; a_1(k), a_0(k) \in K[k], \;\; a_1(k) \perp a_0(k), \quad (8)$$

and tries to construct $r'(k) \in K(k)$ such that

$$a_0(k)r'(k+1) + a_1(k)r'(k) = -a_1(k) \quad (9)$$

(this can also be done by the algorithms from [1] or [2]). If such r' exists then $R = r'$.

If L is as in (8), then $\mathcal{GA}$ succeeds on L iff $\mathcal{AS}$ does: if $\rho = 1$ and $r(k)$ is a rational solution of (4), then (8) has the rational solution

$$r'(k) = -r(k-1)a_1(k-1). \quad (10)$$

In this case both $\mathcal{AS}$ and $\mathcal{GA}$ produce the same operator (rational function) $R = r'$.

Let $\rho \geq 1$ and suppose that there exists

$$r(k) = \frac{s(k)}{q(k)}, \quad s(k) \perp q(k), \tag{11}$$

which satisfies (4). Let R be the result of applying $\mathcal{AS}$ to L of the type (1), and let a polynomial $d \in K[k]$ and an operator $B \in K[k, E]$ with relatively prime coefficients be such that

$$E^\rho \circ L^* \circ \frac{1}{q} = \frac{1}{d} B. \tag{12}$$

Set

$$p(k) = d(k - \rho), \tag{13}$$

$$\bar{L} = B^* \circ E^\rho. \tag{14}$$

Then one gets

$$L \circ p = q\bar{L} \tag{15}$$

and

$$R \circ p \in K[k, E] \tag{16}$$

(this was deduced in [4]).

3 $\mathcal{AS}$ and the discrete Newton-Leibniz formula

The following sufficient condition for validity of (3) is a consequence of Theorem 5 from [4]: If a K-valued sequence $\bar{t}(k)$ is defined and satisfies $\bar{L}\bar{t} = 0$ for all $k \in \mathbb{Z}$, then $t = p\bar{t}$ satisfies $Lt = 0$ for all k, and the discrete Newton-Leibniz formula (3) can be applied to t with $u = Rt = (R \circ p)\bar{t}$ and any integer bounds $v \leq w$. In this section we prove also the necessity of this condition.

Let R be an operator of type (6). We call the monic polynomial

$$\mathrm{den}(R) = \mathrm{lcm}(\mathrm{den}(c_{\rho-1}(k - \rho + 1)), \mathrm{den}(c_{\rho-2}(k - \rho + 2)), \ldots, \mathrm{den}(c_0(k)))$$

the *denominator* of R. It is evident that the operator $R \circ \mathrm{den}(R)$ has polynomial coefficients (i.e., belongs to $K[k, E]$).

In the rest of this paper we suppose that the operator R can be applied to a sequence t only if the sequence t is represented in the form

$$t = \mathrm{den}(R)t', \tag{17}$$

where t' is a sequence defined for all k. In this case we compute the value of Rt for any integer k as the value of the sequence $(R \circ \mathrm{den}(R))t'$. If a sequence $t(k)$ is defined for all k and annihilated by an operator from $K[k, E]$, and if $\mathcal{AS}$ or $\mathcal{GA}$ is applicable to the minimal annihilator of this sequence returning an operator R as result, then t has to be represented in the form (17) before using (3) with $u = Rt$ (in the case where $\mathrm{den}(R)$ has integer zeros, the application of R to t is not possible without such representation).

Certainly, representation (17) does not guarantee that (3) gives the correct result.

Proposition 1 *Let L be of the type (1), $r = \frac{s}{q}$ satisfy (4), and let R satisfy (5). Then* $\mathrm{den}(R) = p$, *where the polynomial p is as in (13).*

Proof: First we show that

$$p \,|\, \mathrm{den}(R). \tag{18}$$

We have

$$E^\rho \circ L^* \circ \frac{1}{q} = \frac{a_0(k+\rho)}{q(k+\rho)}E^\rho + \cdots + \frac{a_{\rho-1}(k+1)}{q(k+1)}E + \frac{a_\rho(k)}{q(k)} \tag{19}$$

(notice that the coefficients of E^i's in the right-hand side of (19) may be reducible). By (12)

$$d(k) \,|\, \mathrm{lcm}\left(\mathrm{den}\left(\frac{a_\rho(k)}{q(k)}\right), \mathrm{den}\left(\frac{a_{\rho-1}(k+1)}{q(k+1)}\right), \ldots, \mathrm{den}\left(\frac{a_0(k+\rho)}{q(k+\rho)}\right)\right).$$

Let $d = d_1^{\alpha_1} \cdots d_m^{\alpha_m}$ be the full factorization of the polynomial d. Then for each i there is an l such that $d_i^{\alpha_i}(k) | \,\mathrm{den}\left(\frac{a_l(k+\rho-l)}{q(k+\rho-l)}\right)$, so let

$$\nu_i = \min\left\{l \ : \ d_i^{\alpha_i}(k) | \,\mathrm{den}\left(\frac{a_l(k+\rho-l)}{q(k+\rho-l)}\right)\right\},$$

for $i = 1, 2, \ldots, m$. Notice that any polynomial $d_i^{\alpha_i}(k)$ divides the denominators of at least two coefficients of the right hand side of (19), since $E^\rho \circ L^*(\frac{s}{q}) = 1 \in K[k]$. This gives us $0 \leq \nu_i \leq \rho - 1$, $i = 1, 2, \ldots, m$. Since $E^\rho \circ L^* \circ r = E^\rho \circ L^* \circ \frac{s}{q}$, and $s \perp q$, we have

$$\nu_i = \min\{l \ : \ d_i^{\alpha_i}(k) | \,\mathrm{den}(a_l(k+\rho-l)r(k+\rho-l))\}, \tag{20}$$

$i = 1, 2, \ldots, m$. Formula (7) is equivalent to

$$c_i(k+\rho-\tau) = \sum_{j=0}^{\tau} r(k+\rho-\tau+j)a_{\tau-j}(k+\rho-\tau+j) - 1$$

for $0 \leq \tau \leq \rho - 1$. If $\tau = \nu_i$, then it follows from this and from (20) that

$$d_i^{\alpha_i}(k)|\mathrm{den}(r(k+\rho-\tau+j)a_{\tau-j}(k+\rho-\tau+j))$$

iff $j = \tau$. As a consequence we have

$$d_i^{\alpha_i}(k)|\mathrm{den}(c_{\nu_i}(k+\rho-\nu_i)),$$

or, equivalently,

$$d_i^{\alpha_i}(k-\rho)|\mathrm{den}(c_{\nu_i}(k-\nu_i)),$$

for $i = 1, 2, \ldots, m$. This implies that

$$d_i^{\alpha_i}(k-\rho) \,|\, \mathrm{den}(R),$$

for all $i = 1, 2, \ldots, m$. Relation (18) follows since $p(k) = d(k-\rho)$.

From (16) it follows that $\mathrm{den}(R)|p$ as well. Since both p and $\mathrm{den}(R)$ are monic, we have $p = \mathrm{den}(R)$. □

Now we can prove the following criterion for validity of the discrete Newton-Leibniz formula in the case where $\mathcal{AS}$ succeeds on a given operator of order $\rho \geq 1$.

Theorem 1 *Let*

- *L be of type (1), a sequence $t(k)$ be defined and $Lt = 0$ for all k,*
- *$r = \frac{s}{q}$, $s \perp q$, satisfy (4), and R be found from (5),*
- *$p, \bar{L}$ be such as in (13), (14),*
- *$\bar{t}(k)$ be a sequence such that $t(k) = p(k)\bar{t}(k)$ for all k.*

Then (3) is applicable everywhere iff $\bar{L}\bar{t}(k) = 0$ for all $k \in \mathbb{Z}$. (If $\bar{L}\bar{t}(k) = 0$ for all $k \in \mathbb{Z}$, then $u = (R \circ p)\bar{t}$ in (3).)

Proof: Let (3) be applicable everywhere with $u = (R \circ p)\bar{t}$. We have from (5):

$$E \circ R - R = 1 - rL, \tag{21}$$

and, as a consequence,

$$E \circ R \circ p - R \circ p - p = -rL \circ p. \tag{22}$$

By (15) we have

$$rL \circ p = rq\bar{L} = \frac{s}{q} q\bar{L} = s\bar{L},$$

therefore

$$E \circ R \circ p - R \circ p - p = -s\bar{L}. \tag{23}$$

Since the sequence $\bar{t}$ is defined for all $k \in \mathbb{Z}$, $(E-1)u = t$, $u = (R \circ p)t$, and $t = p\bar{t}$ for all $k \in \mathbb{Z}$, we have

$$(E \circ R \circ p)\bar{t} - (R \circ p)\bar{t} - p\bar{t} = 0.$$

It follows from (23) that $s\bar{L}\bar{t} = 0$, and if k_0 is such that $s(k_0) \neq 0$ then $\bar{L}\bar{t}(k_0) = 0$ (i.e., the value of the term $\bar{L}\bar{t}$ is equal to 0 when $k = k_0$). If $s(k_0) = 0$, then by $s(k) \perp q(k)$ we have $q(k_0) \neq 0$ and

$$\bar{L}\bar{t}(k_0) = \frac{1}{q(k_0)} Lp(k_0)\bar{t}(k_0) = \frac{1}{q(k_0)} Lt(k_0)$$

as a consequence of (15). However, $Lt = 0$ identically, hence $\bar{L}\bar{t}(k_0) = 0$.

If $\bar{L}\bar{t}(k) = 0$ for all $k \in \mathbb{Z}$ then (3) is applicable everywhere with $u = (R \circ p)\bar{t}$ by Theorem 5 of [4]. □

Example 2 *In Example 6 from [4] the operator* $L = (k-3)(k-2)(k+1)E^2 - (k-3)(k^2-2k-1)E - (k-2)^2$ *was considered to demonstrate some sufficient conditions of applicability of the discrete Newton-Leibniz formula. It was shown, in particular, that* $\mathcal{AS}$ *succeeds on* L *and returns*

$$r = \frac{-1}{(k-2)(k-3)}, \quad R = kE + \frac{1}{k-3}.$$

Apply the criterion from Theorem 1 to L*. We get* $q(k) = (k-2)(k-3)$*,* $p = k-3$*, and*

$$\bar{L} = (k-1)(k+1)E^2 - (k^2-2k-1)E - (k-2).$$

We have $\dim V(\bar{L}) = 2$*, since each of solutions of* $\bar{L}\bar{t} = 0$ *is defined uniquely by* $\bar{t}(2)$ *and* $\bar{t}(3)$ *and by the equation* $\bar{L}\bar{t} = 0$ *when* $k < 2$ *or* $k > 3$*. The sequences* $p(k)\bar{t}_1(k)$*,* $p(k)\bar{t}_2(k)$ *such that* $\bar{t}_1(k), \bar{t}_2(k) \in V(\bar{L})$*,* $\bar{t}_1(2) = 0$*,*

$\bar{t}_1(3) = 1$, $\bar{t}_2(2) = 1$, $\bar{t}_2(3) = 0$ *are linearly independent over* $\mathbb{C}$*: while* $p(2)\bar{t}_1(2) = p(3)\bar{t}_1(3) = 0$*, nevertheless* $p(4)\bar{t}_1(4) = -\frac{1}{3}, p(4)\bar{t}_2(4) = 0$*, and*

$$\begin{vmatrix} p(2)\bar{t}_1(2) & p(4)\bar{t}_1(4) \\ p(2)\bar{t}_2(2) & p(4)\bar{t}_2(4) \end{vmatrix} \neq 0.$$

By our criterion, formula (3) is applicable to $t(k) \in V(L)$ *iff* $t(k) = (k-3)(c_1\bar{t}_1(k) + c_2\bar{t}_2(k))$*,* $c_1, c_2 \in \mathbb{C}$*. Notice that* $\dim V(L) = 3$*: we can take any* $t(2), t(3), t(4), t(5)$ *such that*

$$(k-3)(k-2)(k+1)t(k+2) - (k-3)(k^2-2k-1)t(k+1) - (k-2)^2t(k) = 0$$

for $k = 2, 3$ *(this gives the only constraint* $t(3) = 0$*) and define* $t(k)$ *by the equation* $Lt = 0$ *when* $k < 2$ *or* $k > 5$*.*

4 The case $\mathrm{ord}L = 1$

In the case of $\mathrm{ord}L = 1$ it is possible to prove that $\bar{L}$ and p, defined as in (13), (14), have some additional useful properties. This enables us to simplify the general criterion from Theorem 1.

Proposition 2 *Let* $\rho = 1$ *and* $\bar{L}$ *be as in (14). If* $\bar{L} = \bar{a}_1(k)E + \bar{a}_0(k)$ *then* $\bar{a}_1(k)|a_1(k), \bar{a}_0(k)|a_0(k)$*.*

Proof: It follows from (4) (the case $\rho = 1$), i.e., from

$$a_0(k+1)\frac{s(k+1)}{q(k+1)} + a_1(k)\frac{s(k)}{q(k)} = 1,$$

that the denominators of both terms (after reduction) in the left-hand side are equal:

$$\frac{q(k+1)}{\gcd(a_0(k+1), q(k+1))} = \frac{q(k)}{\gcd(a_1(k), q(k))}. \tag{24}$$

We can compute $p(k)$ using this. Indeed,

$$E \circ L^* \circ \frac{1}{q} = \frac{a_0(k+1)}{q(k+1)}E + \frac{a_1(k)}{q(k)}.$$

Therefore if $d(k) \in K[k]$ and $B \in K[k, E]$ are such that the coefficients of B are relatively prime and

$$E \circ L^* \circ \frac{1}{q} = \frac{1}{d}B$$

then

$$d(k) = \frac{q(k)}{\gcd(a_1(k), q(k))}$$

and

$$p(k) = d(k-1) = \frac{q(k-1)}{\gcd(a_1(k-1), q(k-1))}. \tag{25}$$

By (24), (25) we have

$$L \circ p = a_1(k) \left(\frac{q(k)}{\gcd(a_1(k), q(k))} \right) E + a_0(k) \left(\frac{q(k)}{\gcd(a_0(k), q(k))} \right).$$

The right hand side of this equation can be rewritten as

$$q(k) \left(\frac{a_1(k)}{\gcd(a_1(k), q(k))} E + \frac{a_0(k)}{\gcd(a_0(k), q(k))} \right).$$

Therefore

$$\bar{L} = \frac{a_1(k)}{\gcd(a_1(k), q(k))} E + \frac{a_0(k)}{\gcd(a_0(k), q(k))}. \tag{26}$$

□

Corollary 1 *In the case of* $\mathrm{ord} L = 1$ *the coefficients of* $\bar{L}$ *are relatively prime, and as a consequence, any* K*-valued sequence* $\bar{t}$ *such that* $\bar{L}\bar{t} = 0$ *is a hypergeometric term.*

By (10) the right-hand side of (25) is equal to the denominator of a rational solution $r'(k)$ of equation (9). We have

Corollary 2 *In the case of* $\mathrm{ord} L = 1$ *the polynomial* p *is the denominator of a rational solution of equation (9). When* p *is known,* $\bar{L}$ *can be computed by removing from* $L \circ p$ *the greatest common polynomial factor of its coefficients.*

If $\mathrm{ord} L = 1$ and one uses $\mathcal{GA}$, then Theorem 1 can be reformulated as the following criterion.

Theorem 2 *Let* L *be of type (8), and let* $\frac{f}{p}$*,* $f \perp p$*, be a rational solution of Gosper's equation (9). Then the discrete Newton-Leibniz formula is applicable everywhere to* t*, iff* $t = p\bar{t}$ *for some hypergeometric term* $\bar{t}$ *defined everywhere. If such* $\bar{t}$ *exists, then* $u = f\bar{t}$ *in (3).*

Proof: This follows from Theorem 1 and Corollary 1. □

Example 3 *(Example 1 continued.) We have* $t_2(k) = k\bar{t}_2(k)$*, where*

$$\bar{t}_2(k) = \begin{cases} \frac{(-1)^k}{(-k-1)!}, & \text{if } k < 0, \\ 0, & \text{if } k \geq 0 \end{cases}$$

is a hypergeometric term defined everywhere. We take $u(k) = \bar{t}_2(k)$ *in (3).*

For the sequence $t_1(k)$ *we have* $t_1(k) = k\bar{t}_1(k)$*, where*

$$\bar{t}_1(k) = \begin{cases} 0, & \text{if } k < 0, \\ k!, & \text{if } k > 0. \end{cases}$$

The sequence $\bar{t}_1$ *is not a hypergeometric term for any value of* $\bar{t}_1(0)$*.*

We can summarize Corollaries 1,2 and Theorem 2 as follows:

Corollary 3 *If* L *is of type (8),* $\mathcal{GA}$ *succeeds on* L *and returns* $R \in K(k)$*,* $\mathrm{den}(R) = p$*, then*

$$V_R(L) = p \cdot V(\mathrm{pp}(L \circ p)),$$

where the operator $\mathrm{pp}(L \circ p)$ *is computed by removing from* $L \circ p$ *the greatest common polynomial factor of its coefficients.*

5 Indefinite summable hypergeometric terms which are definite summable by the discrete Newton-Leibniz formula

If an operator L of the form (8) is such that $\mathcal{AS}$ or $\mathcal{GA}$ succeeds on L, then, using Theorem 1, we can describe the space $V_R(L)$: this is the space of sequences of the form $p\bar{t}$, $\bar{L}\bar{t} = 0$.

Proposition 3 *Let* $p, \bar{L}$ *be as in (25), (26). Then there exists a sequence* $\bar{t}$ *which is defined everywhere and is such that* $\bar{L}\bar{t} = 0$ *for all* $k \in \mathbb{Z}$*, and that* $p\bar{t}$ *is a non-zero sequence.*

Proof: By (24), (25) we can write $p(k) = q(k)/\gcd(a_0(k), q(k)$. So by (26), p is relatively prime with both $\bar{a}_1(k-1)$ and $\bar{a}_0(k)$.

If the equation $\bar{a}_1(k-1) = 0$ has integer roots then set k' to be the maximal one. There exists a sequence $\bar{t}$ which is defined everywhere and satisfies $\bar{L}\bar{t} = 0$ for all k, such that $\bar{t}(k') = 1$ (and $\bar{t}(k) = 0$ for all $k < k'$). Then $p\bar{t}$ is not zero at k' because p is relatively prime with $\bar{a}_1(k-1)$. If the equation $\bar{a}_0(k) = 0$ has integer roots then set k'' to be the minimal one. There exists a sequence $\bar{t}$ which is defined everywhere and satisfies $\bar{L}\bar{t} = 0$ for all k, such that $\bar{t}(k'') = 1$ (and $\bar{t}(k) = 0$ for all $k > k''$). Then $p\bar{t}$ is not zero at k'' because p is relatively prime with $\bar{a}_0(k)$. If $\bar{a}_1(k-1)\bar{a}_0(k) \neq 0$ for all integer k, then there exists a sequence $\bar{t}$ which is defined everywhere, and satisfies $\bar{L}\bar{t} = 0$ and $\bar{t}(k) \neq 0$ for all k. It is evident that $p\bar{t}$ is a non-zero sequence. □

As a consequence we get the following theorem.

Theorem 3 *Let $\mathcal{GA}$ succeed on an operator L of type (8), and let $r'(k) = \frac{f}{p}$, $f \perp p$, be a rational solution of Gosper's equation (9). Then there exists a hypergeometric term $\bar{t}$ which is defined everywhere, and is such that the hypergeometric term $t = p\bar{t}$ is not zero, satisfies $Lt = 0$, and formula (3) is valid with $u = f\bar{t}$ for all $v \leq w$.*

It is possible to give examples showing that in some cases $\operatorname{ord} L = 1$, $\dim V_R(L) > 1$.

Example 4 *Let $L = 2(k^2-4)(k-9)E - (2k-3)(k-1)(k-8)$. Then $\mathcal{GA}$ succeeds on L and returns*

$$r'(k) = -\frac{2(k-3)(k+1)}{k-9}.$$

Here $p(k) = k - 9$ and $\bar{L} = 2(k^2-4)E - (2k-3)(k-1)$. Any sequence $\bar{t}$ which satisfies the equation $\bar{L}\bar{t} = 0$ has $\bar{t}(k) = 0$ for $k = 2$ or $k \leq -2$. The values of $\bar{t}(1)$ and $\bar{t}(3)$ can be chosen arbitrarily, and all the other values are determined uniquely by the recurrence $2(k^2-4)\bar{t}(k+1) = (2k-3)(k-1)\bar{t}(k)$. Hence the solution space of $\bar{L}\bar{t} = 0$ has dimension 2; the space of sequences $p\bar{t}$, $\bar{L}\bar{t} = 0$, has dimension 2 too, since $p(1), p(3) \neq 0$.

At the same time, the space $V(L)$ of all solutions of $Lt = 0$ is of dimension 3. Indeed, if $Lt = 0$, then $t(-2) = t(2) = t(9) = 0$. The value $t(k) = 0$ from $k = -2$ propagates to all $k \leq -2$, but on each of the integer intervals $[-1, 0, 1]$, $[3, 4, 5, 6, 7, 8]$ and $[10, 11, \ldots)$ we can choose one value arbitrarily, and the remaining values on that interval are then determined uniquely. A sequence $t \in V(L)$ belongs to $V_R(L)$ iff $22t(10) - 13t(8) = 0$.

Set

$$m = \min(\{\infty\} \cup \{n \in \mathbb{Z} \,:\, a_0(n) = 0\}), \tag{27}$$

$$M = \max(\{-\infty\} \cup \{n \in \mathbb{Z} \ : \ a_1(n-1) = 0\}). \tag{28}$$

If $M < m$, then pick any integer l such that $M \le l \le m$ and then reset $M = m = l$. It is clear that any sequence $t \in V(L)$ is uniquely determined by the vector $(t(m), t(m+1), \ldots, t(M))$, whose entries satisfy the system of algebraic linear equations:

$$a_1(m+i)t(m+i+1) + a_0(m+i)t(m+i) = 0, \quad i = 0, \ldots, M-m-1 \tag{29}$$

(if $m = M$ then $t(m)$ can be chosen arbitrarily).

Using the values m, M we can present a more formal description of our algorithm for constructing a basis of $V_R(L)$, where L of type (8) is such that $\mathcal{GA}$ succeeds on L and returns R. The algorithm starts with computing m, M as above, and $\bar{L} = \bar{a}_1(k) + \bar{a}_0(k)$, $p(k)$ as in Corollaries 2, 3. Then the system of algebraic linear equations with the unknowns $z_m, z_{m+1}, \ldots, z_M$:

$$\bar{a}_1(m+i)z_{m+i+1} + \bar{a}_0(m+i)z_{m+i} = 0, \quad i = 0, \ldots, M-m-1 \tag{30}$$

has to be solved. (Notice that if the vector $(z_m, z_{m+1}, \ldots, z_M)$ satisfies (30), then the vector $(t(m), t(m+1), \ldots, t(M))$, such that $t(m+i) = p(m+i)z_{m+i}$, $i = 0, \ldots, M-m-1$, satisfies (29).) Let the dimension of the solution space of (30) be λ,

$$(z_{1,m}, \ldots, z_{1,M}), \ \ldots \ , (z_{\lambda,m}, \ldots, z_{\lambda,M})$$

be a basis of this space, and the space generated by the vectors

$$(p(m)z_{1,m}, \ldots, p(M)z_{1,M}), \ \ldots \ , (p(m)z_{\lambda,m}, \ldots, p(M)z_{\lambda,M})$$

be of dimension $\mu \le \lambda$ (if p has no root among the numbers $m, m+1, \ldots, M$, then $\mu = \lambda$). W.l.g. we can assume that the vectors

$$(p(m)z_{1,m}, \ldots, p(M)z_{1,M}), \ \ldots \ , (p(m)z_{\mu,m}, \ldots, p(M)z_{\mu,M})$$

are linearly independent. Then we get a basis $p\bar{t}_1, \ldots, p\bar{t}_\mu$ of $V_R(L)$, where the sequence $\bar{t}_i$ is defined by

$$\bar{t}_i(m) = z_{i,m}, \ \ \bar{t}_i(m+1) = z_{i,m+1} \ \ , \ldots, \ \ \bar{t}_i(M) = z_{i,M},$$

and by the equation $\bar{L}\bar{t} = 0$ when $k < m$ or $k > M$.

We finish with the following remark. If we are interested in the applicability of (3) only for the case $k \ge k_0$, where k_0 is a given integer, then we change (27), 28 by

$$m = \min(\{\infty\} \cup \{n \in \mathbb{Z}, n \ge k_0 \ : \ a_0(n) = 0\}),$$

$$M = \max(\{k_0\} \cup \{n \in \mathbb{Z} \ : \ a_1(n-1) = 0\}).$$

If $M < m$, then reset $m = M$. Respectively, if we are interested only in the case $k \leq k_0$, then

$$m = \min(\{k_0\} \cup \{n \in \mathbb{Z} \ : \ a_0(n) = 0\}),$$

$$M = \max(\{-\infty\} \cup \{n \in \mathbb{Z}, n \leq k_0 \ : \ a_1(n-1) = 0\}),$$

and if $M < m$, then reset $M = m$.

If in Examples 1, 3 we are interested only in the case $k \geq 0$, then we get, e.g., that (3) is applicable to $t_1(k)$ when $w \geq v \geq k_0 = 0$ with $u(k) = k!$.

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