A Criterion for the Applicability of Zeilberger's Algorithm to Rational Functions*

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Abstract

We consider the applicability (or terminating condition) of the well-known Zeilberger's algorithm and give the complete solution to this problem for the case where the original hypergeometric term $F(n,k)$ is a rational function. We specify a class of identities $\sum_{k=0}^{n} F(n,k) = 0$, $F(n,k) \in \mathbb{C}(n,k)$, that cannot be proven by Zeilberger's algorithm. Additionally we give examples showing that the set of hypergeometric terms on which Zeilberger's algorithm terminates is a proper subset of the set of all hypergeometric terms, but a super-set of the set of proper terms.

Résumé

Nous considérons l'applicabilité (ou la condition de terminaison) du célèbre algorithme de Zeilberger et nous donnons la solution complète de ce problème dans le cas où le terme hypergéométrique initial $F(n,k)$ est une fonction rationnelle. Nous indiquons une classe d'identités $\sum_{k=0}^{n} F(n,k) = 0, F(n,k) \in \mathbb{C}(n,k)$, qui ne peuvent être démontrées par l'algorithme de Zeilberger. De plus, nous donnons des exemples qui prouvent que l'ensemble des termes hypergéométriques pour

*Supported in part by Natural Sciences and Engineering Research Council of Canada Grant No. CRD215442-98.

lesquels l'algorithme de Zeilberger se termine est un sous-ensemble propre de l'ensemble de tous les termes hypergéométriques mais un super-ensemble de l'ensemble des termes propres.

Keywords: Zeilberger's algorithm; Hypergeometric term; Rational function; Terminating condition; Linear difference and q-difference operators; Z-pair; Decomposition of indefinite sum.

1 Preliminaries

Zeilberger's algorithm [9, 15, 19], also known as the method of *creative telescoping,* is a useful tool for proving identities of the form

$$\sum_{k=-\infty}^{\infty} F(n,k) = f(n) \ , \tag{1}$$

where $F(n,k)$ and $f(n)$ are the given functions. The algorithm, named hereafter as $\mathcal{Z}$, can also be used for proving identities which include definite sums of the forms such as $\sum_{k=0}^{n} F(n,k)$ (see Example 7). Given a function $F(n,k)$ as input, $\mathcal{Z}$ tries to construct for $F(n,k)$ a *Z-pair* (L,G) which consists of a linear difference operator with coefficients which are polynomials in n over $\mathbb{C}$

$$L = a_\rho(n)E_n^\rho + \cdots + a_1(n)E_n^1 + a_0(n)E_n^0 \ , \tag{2}$$

and a function $G(n,k)$ such that

$$LF(n,k) = G(n,k+1) - G(n,k) \ . \tag{3}$$

(E_n is the shift operator w.r.t. n, defined by $E_nF(n,k) = F(n+1,k)$. Similarly E_k is the shift operator w.r.t. k, defined by $E_kF(n,k) = F(n,k+1)$.) Note that the operator L is *k-free.* If such a Z-pair exists, then set $s(n) = \sum_{k=-\infty}^{\infty} F(n,k)$, and by summing (3) over all integer values of k, we obtain the relation $Ls(n) = G(n,\infty) - G(n,-\infty)$. This gives a possibility to establish various properties of $s(n)$, and to prove identities of the form (1). In some particular cases a Z-pair also allows us to find a closed form of $s(n)$ explicitly.

So for a given input $F(n,k)$, $\mathcal{Z}$ is expected to return a Z-pair (L,G) for $F(n,k)$. Note that the algorithm can only be applied to $F(n,k)$ which is a *hypergeometric term* in both arguments, i.e., there exist first order operators $L_1 \in \mathbb{C}[n,k,E_n]$, $L_2 \in \mathbb{C}[n,k,E_k]$ such that $L_1F = L_2F = 0$. It is shown in [19] that if $F(n,k)$ is a hypergeometric term that has a Z-pair $(L, G(n,k))$ then $G(n,k)$ equals the product of a rational function $R(n,k)$ by $F(n,k)$, and thus is also a hypergeometric term. As a consequence, in the case where $F(n,k)$ is a rational function, $G(n,k)$ is also a rational function. It is noteworthy that a Z-pair does not exist for every hypergeometric term (see Example 2). Furthermore, if it exists it is not uniquely defined, for if (L,G) is a Z-pair for $F(n,k)$ and $M \in \mathbb{C}[n,E_n]$, then $(M \circ L, MG)$ is also a Z-pair for $F(n,k)$. It is proven in [19] that if the Z-pairs for $F(n,k)$ exist, then $\mathcal{Z}$ terminates with one of the Z-pairs and the operator L in the returned Z-pair is of minimal possible order. However, it is not necessarily true that one will obtain a linear recurrence of minimal possible order when summing both sides of (3) over k (see [14]).

The question for what hypergeometric terms the Z-pairs do exist is not conclusively answered although a sufficient condition is known. The "fundamental theorem", first proven in [17] (see also [9, 15, 18]), states that a Z-pair exists if $F(n,k)$ is a *proper term*, i.e., it can be written in the form

$$F(n,k) = P(n,k)\,\frac{\prod_{i=1}^{l}(\alpha_i n + \beta_i k + \gamma_i)!}{\prod_{i=1}^{m}(\alpha'_i n + \beta'_i k + \gamma'_i)!} u^n v^k \quad , \tag{4}$$

where $P(n,k) \in \mathbb{C}[n,k]$, $\alpha_i, \beta_i, \alpha'_i, \beta'_i \in \mathbb{Z}$, l, m are nonnegative integers, $\gamma_i, \gamma'_i, u, v \in \mathbb{C}$. (It follows from [18] that $\gamma_i, \gamma'_i, u, v$ may even contain parameters different from n and k.)

It is possible, however, to give an example of a hypergeometric term that is not a proper term but $\mathcal{Z}$ terminates and returns a Z-pair. It is also possible to give an example of a hypergeometric term that is not a proper term either and $\mathcal{Z}$ never terminates. (Sect. 6 is devoted to those examples.) Therefore the set T of hypergeometric terms on which $\mathcal{Z}$ terminates is a proper subset of the set of all hypergeometric terms, but a super-set of the set of proper terms. The complete explicit description of T, we repeat again, is unknown.

In this paper we present the conclusive answer to the question of specifying the class of rational functions $F(n,k)$ that have Z-pairs or, equivalently, the class of rational functions which, when given as input, allow $\mathcal{Z}$ to terminate. (The rational functions are a particular case of hypergeometric terms.) As a consequence, we suggest an improvement to $\mathcal{Z}$. We will describe a class of identities of the form $\sum_{k=0}^{n} F(n,k) = 0$, $F(n,k) \in \mathbb{C}(n,k)$, such that the corresponding rational functions $F(n,k)$ do not have a Z-pair, i.e., these identities cannot be proven using $\mathcal{Z}$. We will also summarize a similar result for the q-difference case [12].

The preliminary publications on this topic have appeared as [4, 5]. In addition to correcting a few minor mistakes, we simplify the proof of Lemma 4 (Sect. 3), clarify and verify the criterion usage (Sect. 4). A new, complete Maple implementation is described (Sect. 5). We also present a similar result for the q-difference case (Sect. 8).

2 Sum of Two Rational Functions

In the subsequent text we will use the following

Lemma 1 *Let there exist* Z*-pairs for* $F_1, F_2 \in \mathbb{C}(n,k)$*. Then there exists a* Z*-pair for* $F = F_1 + F_2$*.*

Proof : Let $L_1, L_2 \in \mathbb{C}[n, E_n], G_1, G_2 \in \mathbb{C}(n,k)$ be such that

$$L_1 F_1 = (E_k - 1)G_1,\ L_2 F_2 = (E_k - 1)G_2\ .$$

Set $L = \operatorname{lclm}(L_1, L_2)$, $L \in \mathbb{C}[n, E_n]$. We have $L = L_1' \circ L_1 = L_2' \circ L_2$ for some $L_1', L_2' \in \mathbb{C}(n)[E_n]$. Then

$$LF = LF_1 + LF_2 = L_1'(L_1F_1) + L_2'(L_2F_2) = L_1'((E_k - 1)G_1) + L_2'((E_k - 1)G_2)\ .$$

Since $E_kE_n = E_nE_k$ and $E_k a(n) = a(n)E_k$ for any $a(n) \in \mathbb{C}(n)$, the operators L_1', L_2' commute with the operator $E_k - 1$. Thus

$$LF = (E_k - 1)(L_1'G_1 + L_2'G_2)\ . \tag{5}$$

Since $L_1'G_1 + L_2'G_2 \in \mathbb{C}(n,k)$, $(L, L_1'G_1 + L_2'G_2)$ is a Z-pair for F. □

In general the operator L constructed above is not of minimal order.

Example 1 Consider the rational function

$$F = F_1 + F_2,\ F_1 = \frac{1}{n+4\,k+2},\ F_2 = \frac{1}{n+4\,k-3}.$$

Applying $\mathcal{Z}$ to F_1 and F_2 results in the Z-pairs (L_1, G_1), (L_2, G_2) for F_1 and F_2, respectively, where L_1, L_2 have the minimal possible orders:

$$(L_1, G_1) = \left(E_n^4 - 1,\ F_1\right),\ (L_2, G_2) = \left(E_n^4 - 1,\ F_2\right).$$

Set $\tilde{L} = \mathrm{lclm}(L_1, L_2) = E_n^4 - 1$. It follows from Lemma 1 that

$$(\tilde{L}, \tilde{G}) = \left(E_n^4 - 1, \frac{1}{n + 4\,k + 2} + \frac{1}{n + 4\,k - 3}\right)$$

is a Z-pair for F. On the other hand, applying $\mathcal{Z}$ to $F = F_1 + F_2$ results in the Z-pair (L, G) where the operator $L = E_n^3 - E_n^2 + E_n - 1$. Notice that in this example, the difference operator $\tilde{L} = \mathrm{lclm}(L_1, L_2)$ in the Z-pair $(\tilde{L}, \tilde{G})$ is *not* of minimal possible order.

3 A Criterion for the Existence of a Z-pair for a Rational Function

The goal of this section is to give a criterion (a necessary and sufficient condition) for a given rational function $F(n, k)$ to have a Z-pair.

For $F(n, k) \in \mathbb{C}(n, k)$, denote $F(n; k)$ as an element of $\mathbb{C}(n)(k)$ (sometimes, when fitting, as an element of the ring $\mathbb{C}(n)[k]$). We also consider polynomials in k whose coefficients are algebraic functions of n, i.e. they are elements of the ring $\overline{\mathbb{C}(n)}[k]$, and denote these polynomials as $p(n; k), q(n; k)$ and so on.

Suppose $F(n, k) \in \mathbb{C}(n, k)$. By applying to $F(n; k)$ any of the algorithms to solve the decomposition problem [1, 3, 16], we can represent $F(n; k)$ in the form

$$F(n; k) = (E_k - 1)S(n; k) + T(n; k)\ \ ,$$

where $S, T \in \mathbb{C}(n)(k)$ are such that the denominator of $T(n; k)$ has the minimal possible degree. For $(E_k - 1)S(n, k)$ we have a Z-pair $(1, S(n, k))$. By Lemma 1 a Z-pair for $F(n, k)$ exists iff a Z-pair for $T(n, k)$ exists. We can represent $T(n, k)$ in the reduced form

$$T(n, k) = \frac{f(n, k)}{g(n, k)}\ \ , \tag{6}$$

where $f(n,k), g(n,k)$ are elements of $\mathbb{C}[n,k]$. By [1], $g(n,k)$ has the following property:

P1. If $p_1(n;k), p_2(n;k)$ are factors of $g(n;k)$ irreducible over $\overline{\mathbb{C}(n)}$ then $p_1(n;k+h) \neq p_2(n;k)$ for all $h \in \mathbb{Z} \setminus \{0\}$.

On the other hand, if $G, V \in \mathbb{C}(n,k)$ are such that $(E_k - 1)G = V$ and

$$V(n,k) = \frac{a(n,k)}{b(n,k)} \ , \tag{7}$$

where $a(n,k), b(n,k)$ are relatively prime elements of $\mathbb{C}[n,k]$, then $b(n,k)$ has the following property:

P2. If $q_1(n;k)$ is a factor of $b(n;k)$ irreducible over $\overline{\mathbb{C}(n)}$ then there exist a factor $q_2(n;k)$ irreducible over $\overline{\mathbb{C}(n)}$ of $b(n;k)$ and a non-zero integer h such that $q_1(n;k+h) = q_2(n;k)$.

Lemma 2 *Let a rational function $T(n,k)$ of the form (6) be such that $g(n,k)$ has property* **P1**. *Let $L \in \mathbb{C}[n, E_n]$ be such that $LT(n,k)$ is of the form (7) and $b(n,k)$ has property* **P2**. *Then for any factor $u(n;k)$ of the polynomial $g(n;k)$ irreducible over $\overline{\mathbb{C}(n)}$ there exist an irreducible factor $v(n;k)$ of $g(n;k)$ (it is possible that $u(n;k) = v(n;k)$) and $j, h \in \mathbb{Z}$, $j > 0$, such that $u(n;k) = v(n+j; k+h)$.*

Proof : Suppose L is of the form (2). Without loss of generality we can assume $a_0(n) \neq 0$. Otherwise, take a new L and $V(n,k)$, namely $E_n^{-\lambda} \circ L$ and $V(n-\lambda, k)$, where λ is the minimal positive integer such that the coefficient of E_n^{λ} in L is not zero. Then $V(n,k)$ is equal to

$$a_\rho(n)T(n+\rho,k) + \cdots + a_0(n)T(n,k), \ \ a_0(n) \text{ is a non-zero polynomial.}$$

Consider the partial fraction decomposition of $T(n;k)$ over $\overline{\mathbb{C}(n)}$. The application of $a_\nu(n)E_n^\nu$, $0 \leq \nu \leq \rho$, to a simple fraction, i.e., a fraction of the form $s(n)/p(n;k)^m$ where $p(n;k)$ is irreducible, $m \geq 1$, gives another simple fraction. Since $u(n;k)$ is an irreducible factor of $g(n;k)$, the decomposition of $T(n;k)$ contains a fraction of the form

$$\frac{s(n)}{u(n;k)^\mu}, \ s(n) \in \overline{\mathbb{C}(n)}, \ \mu \geq 1 \ .$$

If neither the fraction $(a_0(n)s(n))/u(n;k)^\mu$ nor any other fraction with the denominator $u(n;k)^\mu$ is in the decomposition of $LT(n;k)$ then the decomposition of $T(n;k)$ contains a fraction $t(n)/v(n;k)^\mu$ such that $v(n+j;k) = u(n;k)$, where $0 < j \le \rho$ and $a_j(n)$ is a non-zero polynomial. So in this case we get what was claimed.

Suppose that a fraction with the denominator $u(n;k)^\mu$ is in the decomposition of $LT(n;k)$. Since $LT(n,k) = a(n,k)/b(n,k)$ and $b(n,k)$ has property **P2**, the polynomial $b(n;k)$ has a factor $u(n;k-h), h \neq 0$. This implies that the decomposition of $T(n;k)$ contains a fraction of the form $t(n)/u(n-j;k-h)^\tau$, where $\tau > 0$, $j \ge 0$, and E_n^j has a non-zero coefficient in L. Additionally, the denominator of T has property **P1**; therefore, j must be positive. By setting $v(n,k) = u(n-j,k-h)$ we get what was claimed. $\square$

Lemma 3 *Let $g(n,k) \in \mathbb{C}[n,k]$ and for any factor $p_1(n;k)$ of $g(n;k)$ irreducible over $\overline{\mathbb{C}(n)}$ there exist an irreducible factor $p_2(n;k)$ of $g(n;k)$ and $j_1, h_1 \in \mathbb{Z}, j_1 > 0$ such that $p_1(n;k) = p_2(n+j_1;k+h_1)$. Then there exist $J, H \in \mathbb{Z}, J > 0$ such that $p_1(n,k) = p_1(n+J,k+H)$.*

Proof : If $p_1 = p_2$, then take $(J,H) = (j_1,h_1)$ and the claim follows. Otherwise, for any factor $p_1(n;k)$ of $g(n;k)$ irreducible over $\overline{\mathbb{C}(n)}$, there exist $j_1, h_1 \in \mathbb{Z}$, $j_1 > 0$, such that $p_1(n;k) = p_2(n+j_1;k+h_1)$, where $p_2(n;k)$ is a factor of $g(n;k)$ irreducible over $\overline{\mathbb{C}(n)}$. We can continue this process and construct a sequence $p_1(n;k), p_2(n;k), p_3(n;k), \ldots$ of factors of $g(n;k)$ irreducible over $\overline{\mathbb{C}(n)}$ such that for any $l \ge 1$, we have $p_l(n;k) = p_{l+1}(n+j_l;k+h_l)$, $j_l, h_l \in \mathbb{Z}$, $j_l > 0$. Since $g(n;k)$ has only a finite number of irreducible factors, there exists an irreducible factor $p(n;k)$ such that the relation

$$p_\alpha(n;k) = p_\beta(n;k) = p(n;k)$$

holds for some $1 \le \alpha < \beta$. Then for $J = j_\alpha + \cdots + j_{\beta-1}$, $H = h_\alpha + \cdots + h_{\beta-1}$, we have $p(n;k) = p(n+J;k+H)$, $J > 0$; and for $J' = j_1 + \cdots + j_{\alpha-1}$, $H' = h_1 + \cdots + h_{\alpha-1}$, we have $p_1(n,k) = p(n+J',k+H')$, $J', H' \in \mathbb{Z}$, $J' > 0$. Consequently, $p_1(n,k) = p(n+J+J',k+H+H') = p_1(n+J,k+H)$. $\square$

Definition 1 *A polynomial $p(n,k) \in \mathbb{C}[n,k]$ is* integer-linear *if it has the form $an + bk + c$ where $a, b \in \mathbb{Z}$ and $c \in \mathbb{C}$.*

Lemma 4 *Let* $g(n,k) \in \mathbb{C}[n,k]$ *and for any factor* $p_1(n;k)$ *of* $g(n;k)$ *irreducible over* $\overline{\mathbb{C}(n)}$ *there exist* $J, H \in \mathbb{Z}$, $J > 0$, *such that*

$$p_1(n;k) = p_1(n+J;k+H). \tag{8}$$

Then $g(n,k) = c\, p_1(n,k) \cdots p_m(n,k)$, *where* $c \in \mathbb{C}$ *and* $p_1(n,k), \ldots, p_m(n,k)$ *are integer-linear polynomials.*

Proof : Take any factor $p_1(n;k)$ of $g(n;k)$ irreducible over $\overline{\mathbb{C}(n)}$. It follows from (8) that for all $m \in \mathbb{Z}$

$$p_1(n+mJ;k+mH) = p_1(n;k) \tag{9}$$

with $J \neq 0$. Note that $p_1(n;k)$ is linear in k because the coefficient field $\overline{\mathbb{C}(n)}$ is algebraically closed. We can assume $p_1(n;k)$ to be monic. Let

$$p_1(n;k) = k - \varphi(n) \ ,$$

where $\varphi(n)$ is an algebraic function. Assume that 0 is a regular point of $\varphi(n)$ (otherwise substitute n by $n - z_0$ where $z_0 \in \mathbb{C}$ is any arbitrary regular point of $\varphi(n)$). The substitution of $n = k = 0$ into (9) yields

$$p_1(mJ;mH) = p_1(0;0) \ \text{ for all } m \in \mathbb{Z} \ .$$

This implies that $mH - \varphi(mJ)$ has a constant value for all $m \in \mathbb{Z}$ and, as a consequence, that for some $\gamma \in \mathbb{C}$

$$\varphi(mJ) = mH - \gamma \ \text{ for all } m \in \mathbb{Z} \ .$$

Since $\varphi(n)$ is an algebraic function, we have

$$\varphi(n) = \frac{H}{J} n - \gamma \ \text{ for all } n \in \mathbb{C} \ .$$

The last equality means that

$$p_1(n,k) = k - \frac{H}{J} n + \gamma = \frac{1}{J}\left(Jk - Hn + J\gamma\right) \ . \ \Box$$

Theorem 1 *(Criterion for the existence of a* Z*-pair for a rational function.) Let* $F(n,k) \in \mathbb{C}(n,k)$ *be such that*

$$F(n,k) = (E_k - 1)S(n,k) + T(n,k) \ , \tag{10}$$

$S(n,k), T(n,k) \in \mathbb{C}(n,k)$*, and the denominator* $g(n,k)$ *of* $T(n,k)$ *is such that* $\deg_k g(n,k)$ *has the minimal possible value. Then a* Z*-pair for* $F(n,k)$ *exists iff each factor of* $g(n,k)$ *irreducible in* $\mathbb{C}[n,k]$ *is an integer-linear polynomial.*

Proof: The necessary condition follows from Lemmas 2, 3 and 4. Since $(E_k - 1)S(n,k)$ and $T(n,k)$ (which is a proper term) both have Z-pairs, the sufficient condition follows by applying Lemma 1. $\square$

This approach can possibly be applied to develop a criterion that works in the general case of hypergeometric terms in two variables. Note that in [6] the decomposition problem, which is an analogue of (10), for hypergeometric terms was solved. However, no analogue of Lemma 2 was considered in [6].

4 An Algorithm for Using the Criterion

First we consider the question of how to recognize if a given polynomial can be written in the form

$$k + cn + \gamma, \ c \in \mathbb{Q}, \ \gamma \in \mathbb{C}. \tag{11}$$

Lemma 5 *A monic irreducible polynomial* $p(n;k) \in \overline{\mathbb{C}(n)}[k]$ *has the form (11) iff*

$$p(n; k - cn) \in \mathbb{C}[k]. \tag{12}$$

Proof : If $p(n;k)$ has the form (11) then (12) evidently holds. Conversely, if (12) holds, then

$$p(n; k - cn) = \alpha(k - \beta_1) \dots (k - \beta_m), \ \ \alpha, \beta_1, \dots, \beta_m \in \mathbb{C}.$$

This gives us

$$p(n;k) = \alpha(k + cn - \beta_1) \dots (k + cn - \beta_m).$$

Since $p(n;k)$ is monic and irreducible, we get what was claimed. $\square$

Let $w(n,k) \in \mathbb{C}[n,k]$ and $c \in \mathbb{Q}$. Denote by $w_c(n,k)$ the product of all monic irreducible factors of $w(n,k)$ where each factor has the form (11). If there is no such factor, then $w_c(n,k) = 1$. It is evident that $w_c(n,k) \neq 1$ only for a finite set of values of c.

Theorem 2 *Let $w(n,k) \in \mathbb{C}[n,k], \deg_k w(n,k) > 0$. Let $c_0, \ldots, c_m$ be all rational values of c such that $w_c(n,k) \neq 1$. Set $\delta_i = \deg_k w_{c_i}(n,k)$. Then $w(n,k)$ can be represented as a product of integer-linear factors iff*

$$\delta_0 + \cdots + \delta_m = \deg_k w(n,k). \tag{13}$$

Proof : If $w(n,k)$ can be represented in the desired form, then (13) holds since the $w_{c_0}(n,k), \ldots, w_{c_m}(n,k)$ are pairwise relatively prime. If (13) holds, then any irreducible factor $p(n,k)$ of $w(n,k)$ such that $\deg_k p(n,k) > 0$ divides one of the $w_{c_0}(n,k), \ldots, w_{c_m}(n,k)$. This implies that $p(n,k)$ is an integer-linear polynomial. If $\deg_k p(n,k) = 0$ then $p(n,k)$ is evidently integer-linear.

Notice that Lemma 5 gives us a possibility to find $\deg_k w_c(n,k)$ for all $c \in \mathbb{Q}$ such that $w_c(n,k) \neq 1$, and Theorem 2 shows how to use the criterion for an arbitrary rational function. We now describe an algorithm to determine the applicability of $\mathcal{Z}$ to rational functions.

Let $F(n,k)$ be a given rational function. Represent $F(n,k)$ in the form (10) and rewrite $T(n,k)$ as the quotient $f(n,k)/g(n,k)$ of two relatively prime polynomials from $\mathbb{C}[n,k]$. Now we can apply Lemma 5 and Theorem 2 to $g(n,k)$, but to simplify the computation, first extract from $g(n,k)$ the maximal factors $v_1(n) \in \mathbb{C}[n]$ and $v_2(k) \in \mathbb{C}[k]$. Set

$$w(n,k) = g(n,k)/(v_1(n)v_2(k)) \in \mathbb{C}[n,k].$$

Now it remains to investigate whether $w(n,k)$ can be decomposed into factors of the form

$$k + cn + \gamma, \;\; c \in \mathbb{Q} \setminus \{0\}, \; \gamma \in \mathbb{C} \tag{14}$$

or not. Substitute $k - cn$ into $w(n,k)$ for k (this gives us a polynomial $\tilde{w}(c,n,k)$) and compute all nonzero rational values of c such that $\tilde{w}(c,n,k)$ has a non-constant factor from $\mathbb{C}[k]$. To attain this goal we represent $\tilde{w}(c,n,k)$ as a polynomial in n with coefficients in $\mathbb{C}[c,k]$ and find all nonzero rational values of c such that these coefficients have a non-constant greatest common divisor (a polynomial w_c from $\mathbb{C}[k]$ for each value of c). This can be achieved by using resultant or subresultant approaches [2]. We find $c_0, \ldots, c_m$, i.e., all non-zero rational values of c such that $\deg_k w_c(n,k) \neq 0$. Set $\delta_i = \deg_k w_{c_i}(n,k)$. To check whether the criterion holds, it is sufficient to check if relation (13) is satisfied.

Note that the algorithm does not require a complete factorization of the denominator $g(n,k)$ into integer-linear factors.

We conclude this section with a description of the algorithm is$\mathcal{Z}$applicable which determines the applicability of $\mathcal{Z}$ to $F(n,k) \in \mathbb{C}(n,k)$.

`algorithm` is$\mathcal{Z}$applicable;
`input:` a rational function $F(n,k) \in \mathbb{C}(n,k)$;
`output:` $true$ if $\mathcal{Z}$ is applicable to $F(n,k)$; $false$ otherwise;

apply an algorithm to solve the rational sum decomposition
  problem w.r.t. k to obtain $S(n,k), T(n,k)$ in (10);
if $T(n,k) = 0$ **then return** $true$; **fi**;
$f(n,k) :=$ numerator $(T(n,k))$; $g(n,k) :=$ denominator $(T(n,k))$;
$v_1(n) :=$ content$_k$ $(g(n,k))$; $w(n,k) := g(n,k)/v_1(n)$;
$v_2(k) :=$ content$_n$ $(w(n,k))$; $w(n,k) := w(n,k)/v_2(k)$;
if $w(n,k) = 1$ **then return** $true$; **fi**;
$\tilde{w}(c,n,k) := w(n, k - cn)$;
let $\{a_1(c,k), \ldots, a_\rho(c,k)\}$ be the coefficients of $\tilde{w}(c,n,k) \in \mathbb{C}[c,k][n]$;
for $i = 1, 2, \ldots, \rho - 1$ **do**
  for $j = i+1, i+2, \ldots, \rho$ **do**
    $r :=$ resultant$_k$ $(a_i(c,k), a_j(c,k))$;
    if $r \neq 0$ **then**
      let $s = \{c_0, \ldots, c_m\}$ be the non-zero rational roots of r;
      if $s = \{\}$ **then return** $false$; **fi**;
      for $t = 0, 1, \ldots, m$ **do**
        $w_{c_t}(k) :=$ content$_n$ $(\tilde{w}(c_t, n, k))$;
        $\delta_t := \deg_k w_{c_t}(k)$;
      od;
      if $\deg_k w(n,k) = (\delta_0 + \cdots + \delta_m)$ **then**
        return $true$;
      else
        return $false$;
      fi;

fi;
od;
od;

The correctness of the algorithm follows from Lemma 5 and Theorem 2.

5 Implementation

The criterion usage and related functionalities are implemented in Maple 6. They are grouped together into a package, named `Zeilberger`, by using the module-based approach (see Chapter 6, [13]).

`> eval(Zeilberger};`
module Zeilberger ()
export IsHypergeomTerm, SumDecomposition, Gosper, Zeilberger, is_Z_applicable, Z_verify;
option package;
description
"Implementation of Zeilberger's algorithm for the difference case";
end module

The exported local variables indicate the functions that are available. They include:

- IsHypergeomTerm(F, n): check if F is a hypergeometric term in n;
- SumDecomposition(F, n): application of the algorithm to solve the rational sum decomposition problem on F w.r.t. n [3];
- Gosper(F, n): application of Gosper's algorithm on F w.r.t. n;
- Zeilberger(F, n, k, E_n): application of Zeilberger's algorithm on $F(n, k)$;
- is_Z_applicable(F, E_n, n, k): implementation of the criterion usage as described in Sect. 4;
- Z_verify$(F, Z\text{-pair}, E_n, n, E_k, k)$: verification of the result from Zeilberger and is_Z_applicable.

The procedure is_Z_applicable has the following calling sequence

$$\text{is\_Z\_applicable}(F, E_n, n, k, Z\text{-pair});$$

where F is a rational function in n and k, and E_n denotes the shift operator w.r.t. n. The procedure is_Z_applicable returns `false` if F does not satisfy the criterion as stated in Theorem 1; `true` if it does. In this case, if the fifth optional argument Z-pair (which can be any name) is given, it is assigned to the computed Z-pair (L, G) for F.

The program consists of three main steps:

1. *decomposition problem:* rewrite F in the form (10).
2. *applicability of $\mathcal{Z}$:* check whether the denominator of $T(n, k)$ factors into integer-linear polynomials.
3. *creative telescoping:* if the answer in step 2 is positive, then apply the routine `Zeilberger` to $T(n, k)$ starting with order 1 for the difference operator L until $\mathcal{Z}$ terminates. Then use Lemma 1 to obtain a Z-pair for F (see Example 4).

Note that there exist different implementations of $\mathcal{Z}$ [7, 10, 11, 14, 15] such as `zeil` in the package EKHAD [15], and `sumrecursion` in the distributed Maple package *sumtools* [10]. Since the terminating condition that allows a hypergeometric term to have a Z-pair is unknown, a maximum value of the order of the difference operator L in the Z-pair (L, G) needs to be specified in advance (for instance, the default values are 6 for the parameter `MAXORDER` in `zeil`, and 5 for the global parameter `‘sum/zborder‘` in `sumrecursion`). As a consequence, when given a rational function as input, these programs might fail even if a Z-pair exists, i.e., the maximum order of L is not set high enough, or they simply “waste” CPU time trying to find a Z-pair when no such Z-pair exists. Our program, based on Theorem 1, compensates for these weaknesses. It just calls $\mathcal{Z}$ when it is guaranteed that a Z-pair exists, and if that is the case, there is no need to set an upper limit for the order of L.

For the next two examples, the rational function $T(n, k)$ in the decomposition (10) is identical to the given $F(n, k) \in \mathbb{C}(n, k)$. `infolevel` is also used to show the main steps of the algorithms.

Example 2 Consider the rational function

$$F(n, k) = \frac{1}{k^3 - 5\,n\,k^2 - 2\,k^2 + k\,n - 5\,n^2 - 17\,n + 3\,k - 6}\ .$$

The denominator can be written in the form $-(k-5n-2)(k^2+n+3)$. It does not satisfy the criterion, and hence there does not exist any *Z*-pair for F. It takes our program 0.28 seconds to return the desired answer, as opposed to 7382.53 seconds for `zeil` and about 18569 seconds for `sumrecursion` to return the inconclusive answers *"No recurrence of order* ≤ 6 *was found"* and *"System error, ran out of memory"*, respectively [1].

```
> with(Zeilberger);
```

$$[\text{IsHypergeomTerm, SumDecomposition, Gosper, Zeilberger,}\\ \text{is\_Z\_applicable, Z\_verify}]$$

```
> F := 1/(k^3-5*n*k^2-2*k^2+k*n-5*n^2-17*n+3*k-6):
> is_Z_applicable(F,E_n,n,k);
"solve the decomposition problem for the input function"
"check for the applicability of Z"
"Z is not applicable"
```

$$false$$

Example 3 Consider the rational function

$$F(n,k) = \frac{1}{n^2+9\,nk-4\,n-22\,k^2+21\,k-5}\ .$$

The denominator can be written as $(n-2\,k+1)\,(n+11\,k-5)$. Therefore, $F(n,k)$ satisfies the criterion. This example illustrates the case when both `zeil` and `sumrecursion` fail even though a *Z*-pair (L,G) exists. `zeil` returns *"No recurrence of order* ≤ 6 *was found"*, and `sumrecursion` returns *FAIL* (we use the default values of the orders of L for these two programs).

```
> F := 1/(n^2+9*n*k-4*n-22*k^2+21*k-5):
> is_Z_applicable(F,E_n,n,k,'Z_pair');
"solve the decomposition problem for the input function"
"check for the applicability of Z"
"Z is applicable"
"find a Z-pair for the input rational function"
"The computation of a Z-pair is successful"
```

$$true$$

[1]All the reported timings were obtained on 400Mhz, 1Gb RAM, SUN SPARC SOLARIS.

The difference operator L in the computed Z-pair (L, G) is

```
> L := Z_pair[1];
```

$$L := (13\,n + 157)\,E\_n^{12} + (13\,n + 144)\,E\_n^{11} - (13\,n + 14)\,E\_n - (13\,n + 1)$$

As for $G(n,k)$, its representation is too big in size to be shown here. But we can verify that $LF = (E_k - 1)G$:

```
> Z_verify(F,Z_pair,E_n,n,E_k,k);
```

$$true$$

Example 4 In step 3 (creative telescoping) of the algorithm, we suggest that $\mathcal{Z}$ be applied to $T(n,k)$ and then Lemma 1 be used to obtain the computed Z-pair, as opposed to applying $\mathcal{Z}$ directly to the input rational function. (It is easy to check that the application of Lemma 1 in this case does give the operator L in (5) of minimal possible order.) Let us name our algorithm *$\mathcal{Z}$-modified*, and the classical $\mathcal{Z}$ *$\mathcal{Z}$-original*. We now compare the two algorithms via a set of examples where $S(n,k)$ in the decomposition (10) is non-trivial (the cost is the same otherwise).

Set

$$T(n,k) = \frac{8\,n - 7\,k - 4}{(k-3)^2\,(k+n-5)^3}$$

in (10). Table 2 shows the timing (in seconds) and memory (in bytes) required by the two algorithms on a set of examples where $S_{i,j}(n,k)$ are randomly generated (see Table 1; the indices i, j denote the total degrees of the numerator and the denominator of $S(n,k)$, resp.). It also shows the speedup factors and the reductions in memory usage when $\mathcal{Z}$-modified is used. The results were verified by using the routine `Z_verify`.

Table 1: The set of randomly-generated $S_{i,j}(n,k)$ used for testing.

$S_{1,1} = (4 - k - 4n)/(-1 - k - 4n)$
$S_{1,2} = (-2 + 3k + 4n)/(4 - 3k - 3n + 4nk - n^2)$
$S_{1,3} = (5 + k + 5n)/(-n - nk - 2k^2 - 3nk^2 + 3k^3 - 5n^2k)$
$S_{2,1} = (5 + 2k + 4n - 3nk - 3n^2 - k^2)/(-5 - 2k - 2n)$
$S_{2,2} = (-4 + k + n + 3nk - 4n^2)/(4 - k + 2n + 4nk + n^2 + 3k^2)$
$S_{2,3} = (-5 + 3n - 5nk + n^2 - 4k^2)/(2nk + 3k^2 + nk^2 + 3n^3 + 5n^2k)$

Table 2: Time and space requirements of $\mathcal{Z}$-original and $\mathcal{Z}$-modified.

	Timing			Memory		
	$\mathcal{Z}$-original	$\mathcal{Z}$-modified	Speedup	$\mathcal{Z}$-original	$\mathcal{Z}$-modified	Reduction
$S_{1,1}$	45.580	5.570	8.183	168,974,832	25,003,012	85.20%
$S_{1,2}$	83.170	6.290	13.220	316,153,824	26,150,832	91.73%
$S_{1,3}$	310.510	8.380	37.053	720,545,420	30,709,100	95.74%
$S_{2,1}$	12.110	5.970	2.028	49,911,296	23,388,768	53.14%
$S_{2,2}$	87.980	6.390	13.770	323,202,104	25,854,140	92.00%
$S_{2,3}$	305.350	7.860	38.850	908,073,148	31,811,268	96.50%

Example 5 We now show an example of a sequence of rational functions $F_0(n,k)$, $F_1(n,k),\ldots$ such that a Z-pair (L_m, G_m) for $F_m(n,k)$ exists for every $m \in \mathbb{N}$, and $\operatorname{ord} L_m > m$, i.e., it is not always possible to set the order of L high enough.

Consider the sequence of rational functions

$$F_m(n,k) = \frac{1}{n+(m+1)k}, \quad m \in \mathbb{N} \ .$$

It is easy to check that $(L_m, G_m) = (E_n^{m+1} - 1, F_m)$ is a Z-pair for F_m. Notice that $\operatorname{ord} L_m = m+1 > m$. Suppose there exists $L'_m \in \mathbb{C}[n, E_n]$ such that $\operatorname{ord} L'_m \leq m$ and $L'_m F_m = (E_k - 1)G'_m$ for some $G'_m \in \mathbb{C}(n,k)$. We can assume that the coefficient of E_n^0 in L'_m is a non-zero element of $\mathbb{C}[n]$. Otherwise, choose the new Z-pair for F_m

$$(E_n^{-\lambda} \circ L'_m, G'_m(n-\lambda, k))$$

where λ is the minimal positive integer such that the coefficient of E_n^λ in L'_m is not zero. Set $H_m = L'_m F_m = a(n,k)/b(n,k)$ taken in reduced form. Since H_m is rational summable, $b(n,k)$ has property **P2**. Therefore, for the factor $n + k(m+1)$ of $b(n,k)$, there exists a non-zero integer h such that $n+(k+h)(m+1)$ is also a factor of $b(n,k)$. Since all the irreducible factors of $b(n,k)$ have the form $n+i+k(m+1)$, $i = 0, 1, \ldots, \operatorname{ord} L'_m$, this means $(n+(k+h)(m+1)) - (n+i+k(m+1)) = h(m+1) - i$ is the zero polynomial for some i. This is not possible since $0 \leq i \leq m$ and $h \neq 0$.

6 A Remark on Zeilberger's Algorithm and Proper Terms

It is not easy to find in the literature an example of a hypergeometric term to which $\mathcal{Z}$ is not applicable. For instance, the book [15], especially devoted to certifying identities, does not have such an example. In [9] (p. 239), first the very true statement that $\mathcal{Z}$ occasionally does not work is given. The authors then state that $\mathcal{Z}$ fails on the simple hypergeometric term $1/(nk+1)$ and refer the readers to Ex. 107. This exercise (p. 255) asks to prove that $1/(nk+1)$ is not a proper hypergeometric term. But the fact that a hypergeometric term is not proper does not imply that $\mathcal{Z}$ fails on that hypergeometric term (see Example 6 below). In a similar manner it is shown in [18] that $1/(n^2+k^2)$ is not a holonomic function (see [18] for the definition) since there does not exist any annihilator from $\mathbb{C}[n, E_n, E_k]$ for $1/(n^2+k^2)$ (it was proven preliminarily that for any holonomic function such an annihilator must exist). But, again, this does not give grounds for claiming that $\mathcal{Z}$ fails on $1/(n^2+k^2)$.

Based on the criterion established in Sect. 3, it is clear that there does not exist any Z-pair for $1/(nk+1)$ and $1/(n^2+k^2)$. Hence $\mathcal{Z}$ fails on them (a direct short proof that $1/(nk+1)$ does not have any Z-pair is presented in [4]). It is also clear from Example 6 that the non-existence of an annihilator from $\mathbb{C}[n, E_n, E_k]$ for a given hypergeometric term does not imply that $\mathcal{Z}$ fails on this hypergeometric term or, equivalently, that there does not exist a Z-pair for this hypergeometric term.

Example 6 Consider

$$F(n,k) = (E_k - 1)\frac{1}{nk+1} = \frac{1}{n(k+1)+1} - \frac{1}{nk+1} \ . \tag{15}$$

It is easy to see that $(1, 1/(nk+1))$ is a Z-pair for the rational function (15). Therefore $\mathcal{Z}$ is applicable to $F(n,k)$. Now we prove that the hypergeometric term $F(n,k)$ is not proper. Although (15) is not written in proper hypergeometric form (4), we do not have yet any argument to claim that it is not proper. This problem is not so simple: a remark from [9] especially emphasizes that the hypergeometric terms $1/(nk)$ and $1/(n^2-k^2)$ are proper while $1/(nk+1)$ and $1/(n^2+k^2)$ are not. It was proven in [18] (see also [9, 15]) that any proper hypergeometric term can be annihilated by a non-zero operator $M \in \mathbb{C}[n, E_n, E_k]$ (the coefficients depend only on n). It was shown in the solution of Ex. 107 in [9] that for the hypergeometric term $1/(nk+1)$ such M does not exist (it follows that $1/(nk+1)$ is not proper). Suppose that $F(n,k)$ of the form (15) is proper. Then $MF(n,k) = 0$ for some non-zero $M \in \mathbb{C}[n, E_n, E_k]$ and hence

$$M((E_k - 1)\frac{1}{nk+1}) = (M \circ (E_k - 1))\frac{1}{nk+1} = 0 \ .$$

But $M \circ (E_k - 1)$ is a non-zero operator from $\mathbb{C}[n, E_n, E_k]$. Contradiction.

So it is not true that $\mathcal{Z}$ is applicable to all rational functions. It is also not true that $\mathcal{Z}$ is applicable to a rational function $F(n,k)$ only if $F(n,k)$ is a proper term. Finally, the non-existence of an annihilator from $\mathbb{C}[n, E_n, E_k]$ for a given rational function $F(n,k)$ does not in general imply that $\mathcal{Z}$ fails on $F(n,k)$.

7 On a Class of Evident Identities

Suppose $R(n,k)$ is a rational function that has no pole at (n_0, k_0) with $n_0, k_0 \in \mathbb{Z}$, $0 \le k_0 \le n_0$. Then clearly

$$\sum_{k=0}^{n} F(n,k) = 0 \ , \tag{16}$$

where

$$F(n,k) = R(n,k) - R(n, n-k) \ . \tag{17}$$

If there exists a Z-pair for (17), we can use $\mathcal{Z}$ to prove identities of the form (16).

Example 7 Let $R(n,k) = 1/(k+1)$ and, resp., $F(n,k) = 1/(k+1) - 1/(n-k+1)$. Then $F(n,k)$ has a Z-pair $(E_n - 1, 1/(n-k+2))$:

$$F(n+1,k) - F(n,k) = \frac{1}{n-k+1} - \frac{1}{n-k+2} \ .$$

By applying the summation operator $\sum_{k=0}^{n}$ to both sides of the last equality, we obtain

$$\sum_{k=0}^{n} F(n+1,k) - \sum_{k=0}^{n} F(n,k) = 1 - \frac{1}{n+2} \ . \tag{18}$$

Set $s(n) = \sum_{k=0}^{n} F(n,k)$. We have from (18) that $s(n+1) - s(n) = 0$. This difference equation is of order 1 and its leading coefficient does not vanish when $n \geq 0$. Therefore it is sufficient to check (16) for $n = 0$. The result of this checking is positive, i.e., $s(0) = 0$. However this method of identity proving is possible only if the given rational function $F(n,k)$ satisfies the criterion formulated above. A rational function of the form (17) in most cases does not have a Z-pair and $\mathcal{Z}$ fails on this function. This takes place, for instance, if $F(n,k) = R(n,k) - R(n, n-k)$ and $R(n,k)$ is one of the following rational functions:

$$\frac{1}{nk+1}, \frac{1}{nk+2}, \ldots$$

or

$$\frac{1}{n^2+k^2+1}, \frac{1}{n^2+k^2+2}, \ldots$$

or

$$\frac{1}{n^2+k+1}, \frac{1}{n^2+k+2}, \ldots$$

and so on.

8 q-Difference Case

Zeilberger's algorithm can be carried over to the q-difference case [18, 11]. It is shown in [12] that after establishing the q-analogue of Properties **P1** and **P2** of the decomposition problem [3] as described in Sect. 2, one can derive an analogous theorem for the applicability of Zeilberger's algorithm to rational functions in the q-difference case.

Theorem 3 *(Criterion for the existence of a qZ-pair for a rational function.) Let $F(q^n, q^k) \in \mathbb{C}(q)(q^n, q^k)$ be such that*

$$F(q^n, q^k) = (Q_k - 1)S(q^n, q^k) + T(q^n, q^k),$$

$S(q^n, q^k), T(q^n, q^k) \in \mathbb{C}(q)(q^n, q^k)$, and the denominator $g(q^n, q^k)$ of $T(q^n, q^k)$ is such that $\deg_{q^k} g(q^n, q^k)$ has the minimal possible value. Then a qZ-pair for $F(q^n, q^k)$ exists iff

$$g(q^n, q^k) = \alpha q^{an} \prod_i \left(q^k - \gamma_i q^{c_i n}\right), \quad c_i \in \mathbb{Q}, \gamma_i, \alpha \in \overline{\mathbb{C}(q)}, a \in \mathbb{Z}.$$

Note that q is an indeterminate parameter, Q_n, Q_k denote the q-shift operators w.r.t. q^n and q^k, resp., defined by $Q_n F(q^n, q^k) = F(q^{n+1}, q^k)$, $Q_k F(q^n, q^k) = F(q^n, q^{k+1})$.

9 Availability

The Maple package *Zeilberger* and related documents are available and can be downloaded at the following URL

`http://www.scg.uwaterloo.ca/~hqle/Zeilberger/difference/`.

Acknowledgements

We would like to thank Marko Petkovšek of University of Ljubljana for his useful comments on an earlier draft, and Keith Geddes of University of Waterloo for his support.

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