Convolutions of Liouvillian Sequences

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Abstract

While Liouvillian sequences are closed under many operations, simple examples show that they are not closed under convolution, and the same goes for d'Alembertian sequences. Nevertheless, we show that d'Alembertian sequences are closed under convolution with *rationally* d'Alembertian sequences, and that Liouvillian sequences are closed under convolution with *rationally* Liouvillian sequences.

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1. Introduction

Liouvillian sequences constitute a large class of sequences which can be represented explicitly (as opposed to recursively, by generating functions, or by other means). They were defined in [7] as the elements of the least ring of (germs of) sequences that contains all hypergeometric sequences and is closed under shift, inverse shifts, indefinite summation, and interlacing. In the same paper, an algorithm was given for finding all Liouvillian solutions of linear recurrences with polynomial coefficients. It is interesting to note that the ring of Liouvillian sequences is closed under many operations, but it is not closed under *convolution* or *Cauchy product* of sequences, an important operation which corresponds to the product of their respective (ordinary) generating functions.

Example 1. *Zeilberger's Creative Telescoping algorithm* [14, 15] *establishes that the convolution of* $n!$ *with* $1/n!$

$$y_n \;:=\; n! * \frac{1}{n!} \;=\; \sum_{k=0}^{n} \frac{k!}{(n-k)!}$$

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satisfies the inhomogeneous recurrence

$$y_{n+2} - (n+2)y_{n+1} + y_n = \frac{1}{(n+2)!},$$

which can be homogenized to

$$(n+3)y_{n+3} - (n^2+6n+10)y_{n+2} + (2n+5)y_{n+1} - y_n = 0. \qquad (1)$$

As the Hendriks-Singer algorithm [7] *shows, this recurrence has no nonzero Liouvillian solutions. So the convolution of Liouvillian (even hypergeometric!) sequences* $n!$ *and* $1/n!$ *is not Liouvillian.*

This observation opens several new directions of investigation. One, which is not pursued here, is algorithmic: Design algorithms for finding those solutions of linear recurrences with polynomial coefficients that belong to the least ring of sequences which contains all hypergeometric sequences and is closed under shift, inverse shifts, indefinite summation, interlacing, and convolution.

Another one, which is the focus of this paper, is finding some restrictions on Liouvillian sequences a and/or b which will guarantee that their convolution $a * b$ is Liouvillian. Such results can be used for simplification of expressions containing convolutions of these "restricted Liouvillian" sequences, as they allow us to eliminate convolution and express it by the operations that are used to define Liouvillian sequences. They also show that by finding all Liouvillian solutions of a linear recurrence with polynomial coefficients, we also find all solutions which are convolutions of "restricted Liouvillian" sequences.

The restrictions that we use concern the set of the basis sequences used in the definition of the ring of Liouvillian sequences. A sequence a is *rational* if there is a rational function $r(x)$ such that $a_n = r(n)$ for all large enough n. A sequence a is *quasi-rational* [1] if there are $d \in \mathbb{N}$, rational functions $r_1(x), r_2(x), \ldots, r_d(x)$, and constants $\alpha_1, \alpha_2, \ldots, \alpha_d$ such that $a_n = \sum_{i=1}^{d} r_i(n)\alpha_i^n$ for all large enough n. This is a natural generalization of C-finite (a.k.a. C-recursive) sequences which are solutions of linear recurrences with constant coefficients. We define *(quasi)-rationally Liouvillian sequences* (Def. 5 on p. 5) analogously to Liouvillian sequences, except that the rôle of basis sequences is played by (quasi)-rational sequences rather than by hypergeometric sequences.

One of the main results of the paper is the following: *If* a *is Liouvillian and* b *is quasi-rationally Liouvillian (or vice versa), then* $a*b$ *is Liouvillian* (Theorem 2 on p. 5). The proof is based on the analogous result for d'Alembertian sequences (introduced in [3]) which are the elements of the least ring of sequences that contains all hypergeometric sequences and is closed under shift, inverse shifts, and indefinite summation. We define *(quasi)-rationally d'Alembertian sequences* (Def. 4 on p. 5) analogously to d'Alembertian sequences, except that again the basis sequences are (quasi)-rational sequences rather than hypergeometric ones. Then we have the following result: *If* a *is d'Alembertian and* b *is quasi-rationally d'Alembertian (or vice versa), then* $a * b$ *is d'Alembertian* (Theorem 1 on p. 5). These results, together with the observation that Liouvillian sequences are not closed under convolution (Example 1 on p. 1), appear to be new. Their proofs, given in Sections 5 and 6, are conceptually simple but technically nontrivial because convolution is not a local operation, meaning that equivalent factors need not yield equivalent results (see Defs. 6 and 7 on p. 6). Consequently, we cannot use the ring of germs of sequences which has a nicer algebraic structure, but are forced to work with sequences themselves, ensuring that each term of every sequence that we consider is well defined. The proofs of Theorems 1 resp. 2 are constructive and can be used to convert the convolution of a d'Alembertian sequence with a quasi-rationally d'Alembertian sequence into a standard representation of a d'Alembertian sequence, resp. the convolution of a Liouvillian sequence with a quasi-rationally Liouvillian sequence into a standard representation of a Liouvillian sequence (see Examples 6 resp. 7).

A short overview of the paper: In Section 2 we define the relevant sequence classes and operations, and introduce the corresponding notation (cf. [9]). In Section 3 we state our main results and give some examples. Section 4 contains some technical results concerning closure properties of sequence classes under equivalence [12], (non)-locality of operations, and the relation between convolutions of pairwise equivalent factors. Section 5 begins by listing some of the properties of d'Alembertian sequences needed to prove Theorem 1. The proof itself is divided into two parts: Proposition 5 deals with the "ideal" case where the minimal annihilators of the hypergeometric resp. rational sequences in the two factors are nonsingular, while the rest of the proof takes care of the general case. The section ends by elaborating Example 4. Section 6 contains the proof of Theorem 2 and elaborates Example 5.

The preliminary version [5] of this paper contains some additional results and examples.

2. Preliminaries

Let $\mathbb{K}$ be an algebraically closed field of characteristic 0, $\mathbb{N}$ the set of nonnegative integers, and $(\mathbb{K}^{\mathbb{N}}, +, \cdot)$ the ring of all sequences with terms in $\mathbb{K}$.

Definition 1. *A sequence $\langle a_n\rangle_{n=0}^{\infty} \in \mathbb{K}^{\mathbb{N}}$ is:*

- rational *if there is $r \in \mathbb{K}(x)$ such that $a_n = r(n)$ for all large enough n,*
- quasi-rational (*cf.* [1]) *if there are $d \in \mathbb{N}$, $r_1, r_2, \ldots, r_d \in \mathbb{K}(x)^*$, and $\alpha_1, \alpha_2, \ldots, \alpha_d \in \mathbb{K}^*$ such that $a_n = \sum_{i=1}^{d} r_i(n)\alpha_i^n$ for all large enough n,*
- hypergeometric *if there are $p, q \in \mathbb{K}[x] \setminus \{0\}$ such that*

$$q(n)\, a_{n+1} + p(n)\, a_n \;=\; 0 \text{ for all } n \in \mathbb{N}$$

and $a_n \neq 0$ for all large enough n,

- P-recursive *or* holonomic *if there are* $d \in \mathbb{N}$ *and* $p_0, p_1, \ldots, p_d \in \mathbb{K}[x]$, $p_d \neq 0$, *such that*

$$p_d(n)a_{n+d} + p_{d-1}(n)a_{n+d-1} + \cdots + p_0(n)a_n \;=\; 0$$

for all $n \in \mathbb{N}$.

Additional notation:

- We denote the set of hypergeometric sequences in $\mathbb{K}^{\mathbb{N}}$ by $\mathcal{H}(\mathbb{K})$, and the set of P-recursive sequences in $\mathbb{K}^{\mathbb{N}}$ by $\mathcal{P}(\mathbb{K})$.
- For $n, m \in \mathbb{N}$, $m \geq 1$, we denote by $n \operatorname{div} m := \lfloor \frac{n}{m} \rfloor$ the quotient, and by $n \bmod m := n - m\lfloor \frac{n}{m} \rfloor$ the remainder in integer division of n by m.
- The *shift operator* $E: \mathbb{K}^{\mathbb{N}} \to \mathbb{K}^{\mathbb{N}}$ is defined for all $a \in \mathbb{K}^{\mathbb{N}}$ and $n \in \mathbb{N}$ by $E(a)_n := a_{n+1}$, and for $k \in \mathbb{N}$, its k-fold composition with itself is denoted by E^k. For $d \in \mathbb{N}$ and $p_0, p_1, \ldots, p_d \in \mathbb{K}[x]$ such that $p_d \neq 0$, the map $L = \sum_{k=0}^{d} p_k(n)\, E^k : \mathbb{K}^{\mathbb{N}} \to \mathbb{K}^{\mathbb{N}}$ is a *linear recurrence operator* of order $\operatorname{ord} L = d$ with polynomial coefficients. We denote the Ore algebra of all such operators (with composition as multiplication) by $K[n]\langle E \rangle$.
- The *inverse shift operator* $E_{\lambda}^{-1}: \mathbb{K}^{\mathbb{N}} \to \mathbb{K}^{\mathbb{N}}$ is defined for all $\lambda \in \mathbb{K}$, $a \in \mathbb{K}^{\mathbb{N}}$, and $n \in \mathbb{N}$ by $E_{\lambda}^{-1}(a)_n := \begin{cases} a_{n-1}, & n \geq 1, \\ \lambda, & n = 0. \end{cases}$
- The *indefinite* (or: *partial*) *summation operator* $\Sigma : \mathbb{K}^{\mathbb{N}} \to \mathbb{K}^{\mathbb{N}}$ is defined for all $a \in \mathbb{K}^{\mathbb{N}}$ and $n \in \mathbb{N}$ by $\Sigma a_n := \sum_{k=0}^{n} a_k$.
- The *multisection operator* $\mu_{m,r} : \mathbb{K}^{\mathbb{N}} \to \mathbb{K}^{\mathbb{N}}$ is defined for all $m \in \mathbb{N} \setminus \{0\}$, $r \in \{0, 1, \ldots, m-1\}$, $a \in \mathbb{K}^{\mathbb{N}}$ and $n \in \mathbb{N}$ by $\mu_{m,r}(a)_n = a_{mn+r}$ *(the r-th m-section of a).*
- The *convolution operator* $* : \mathbb{K}^{\mathbb{N}} \times \mathbb{K}^{\mathbb{N}} \to \mathbb{K}^{\mathbb{N}}$ is defined for all $a, b \in \mathbb{K}^{\mathbb{N}}$ and $n \in \mathbb{N}$ by $(a * b)_n := \sum_{k=0}^{n} a_k b_{n-k}$.
- The *interlacing operator* $\Lambda : \bigcup_{m=1}^{\infty} \left(\mathbb{K}^{\mathbb{N}}\right)^m \to \mathbb{K}^{\mathbb{N}}$ is defined for all $m \geq 1$, $a^{(0)}, a^{(1)}, \ldots, a^{(m-1)} \in \mathbb{K}^{\mathbb{N}}$ and $n \in \mathbb{N}$ by $\Lambda(a^{(0)}, a^{(1)}, \ldots, a^{(m-1)})_n = \left(\Lambda_{j=0}^{m-1} a^{(j)}\right)_n := a_{n \operatorname{div} m}^{(n \bmod m)}$.

Definition 2. *The ring of* d'Alembertian sequences $\mathcal{A}(\mathbb{K})$ *(cf. [3]) is the least subring of* $\mathbb{K}^{\mathbb{N}}$ *which contains* $\mathcal{H}(\mathbb{K})$ *and is closed under shift, all inverse shifts, and indefinite summation.*

Definition 3. *The ring of* Liouvillian sequences $\mathcal{L}(\mathbb{K})$ *(cf. [7]) is the least subring of* $\mathbb{K}^{\mathbb{N}}$ *which contains* $\mathcal{H}(\mathbb{K})$ *and is closed under shift, all inverse shifts, indefinite summation, and interlacing.*

It is well known that $\mathcal{P}(\mathbb{K})$ is closed under shift, all inverse shifts, indefinite summation, multisection, scalar multiplication, addition, multiplication, convolution, and interlacing (cf. [7, 12, 13]). Consequently, $\mathcal{H}(\mathbb{K}) \subseteq \mathcal{A}(\mathbb{K}) \subseteq \mathcal{L}(\mathbb{K}) \subseteq \mathcal{P}(\mathbb{K})$.

As shown by Example 1, neither $\mathcal{A}(\mathbb{K})$ nor $\mathcal{L}(\mathbb{K})$ is closed under convolution. To obtain positive results, we define some subrings of these rings by replacing hypergeometric sequences with (quasi)-rational sequences as their basis.

Definition 4. *The ring of* (quasi)-rationally d'Alembertian sequences $\mathcal{A}_{(q)rat}(\mathbb{K})$ *is the least subring of* $\mathbb{K}^{\mathbb{N}}$ *which contains all (quasi)-rational sequences over* $\mathbb{K}$ *and is closed under shift, all inverse shifts, and indefinite summation.*

Example 2. *Derangement numbers* $d_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$ *are d'Alembertian but not rationally d'Alembertian; harmonic numbers* $H_n = \sum_{k=1}^{n} \frac{1}{k} = \sum_{k=0}^{n} \frac{1}{k+1} - \frac{1}{n+1}$ *are rationally d'Alembertian.*

Definition 5. *The ring of* (quasi)-rationally Liouvillian sequences $\mathcal{L}_{(q)rat}(\mathbb{K})$ *is the least subring of* $\mathbb{K}^{\mathbb{N}}$ *which contains all (quasi)-rational sequences over* $\mathbb{K}$ *and is closed under shift, all inverse shifts, indefinite summation, and interlacing.*

Example 3. *The sequence* $n!!$ *is the interlacing of hypergeometric sequences* $2^n n!$ *and* $(2n+1)!/(2^n n!)$*, hence it is Liouvillian (but it is not rationally Liouvillian). Any interlacing of rationally d'Alembertian sequences is rationally Liouvillian.*

3. Main results

Theorem 1. *If* $a \in \mathbb{K}^{\mathbb{N}}$ *is d'Alembertian and* $b \in \mathbb{K}^{\mathbb{N}}$ *is (quasi)-rationally d'Alembertian, then their convolution* $a * b$ *is d'Alembertian.*

The proof is given in Section 5 on p. 14.

Example 4. *Let* $a_n = 2^{n-1} n!$ *(a hypergeometric sequence) and* $b_n = \dfrac{1}{n+\frac{1}{2}}$ *(a rational sequence). As shown in Example 6 on p. 15, their convolution equals*

$$y_n \;=\; (2n+1)!! \left(1 + \sum_{k=1}^{n} \frac{1}{(2k+1)!!} \left(\frac{1}{2k+1} - \sum_{j=0}^{k-1} (2j)!!\right)\right).$$

As $(2n+1)!!$ *and* $(2n)!!$ *are hypergeometric,* y *is d'Alembertian.*

Theorem 2. *If* $u \in \mathbb{K}^{\mathbb{N}}$ *is Liouvillian and* $v \in \mathbb{K}^{\mathbb{N}}$ *is (quasi)-rationally Liouvillian, then their convolution* $u * v$ *is Liouvillian.*

The proof is given in Section 6 on p. 18.

Example 5. *Let $a_n = n!!$ (a Liouvillian sequence which is not d'Alembertian) and $b_n = \frac{1}{n+1}$ (a rational sequence). As shown in Example 7 on p. 19, their convolution y is the interlacing of $g^{(0)}$ and $g^{(1)}$ where*

$$g_n^{(0)} = (2n+1)!! \left(1 + \sum_{k=1}^{n} \frac{1}{(2k+1)!!} \left(\frac{4k+1}{2k(2k+1)} - \sum_{j=0}^{2k-2} j!!\right)\right) \tag{2}$$

$$g_n^{(1)} = (2n+2)!! \left(\frac{3}{4} + \sum_{k=1}^{n} \frac{1}{(2k+2)!!} \left(\frac{4k+3}{(2k+1)(2k+2)} - \sum_{j=0}^{2k-1} j!!\right)\right) \tag{3}$$

As $\sum_{j=0}^{2k-2} j!! = \sum_{j=0}^{k-1}(2j)!! + \sum_{j=0}^{k-2}(2j+1)!!$ and $\sum_{j=0}^{2k-1} j!! = \sum_{j=0}^{k-1}(2j)!! + \sum_{j=0}^{k-1}(2j+1)!!$, both $g^{(0)}$ and $g^{(1)}$ are d'Alembertian, hence y is Liouvillian.

4. Equivalence of sequences and (non)-locality of operations

Definition 6. [12] *Sequences $a, b \in \mathbb{K}^{\mathbb{N}}$ are* equivalent *if there is an $N \in \mathbb{N}$ s.t.*

$$a_n = b_n \quad \text{for all } n \geq N$$

or equivalently, s.t. $E^N(a) = E^N(b)$. We denote this relation by $\sim$, and call its equivalence classes germs at ∞ of functions $\mathbb{N} \to \mathbb{K}$.

Definition 7.
- *A set of sequences $\mathcal{C} \subseteq \mathbb{K}^{\mathbb{N}}$ is* closed under equivalence *if $a \in \mathcal{C}$ and $a \sim a'$ implies that $a' \in \mathcal{C}$.*
- *An operation ω on $\mathbb{K}^{\mathbb{N}}$ will be called* local *if $\sim$ is a congruence w.r.t. ω (i.e., if equivalent operands produce equivalent results).*

Proposition 1. *The set $\mathcal{H}(\mathbb{K})$ is closed under equivalence.*

Proof: Assume that $a \in \mathcal{H}(\mathbb{K})$ and $a' \sim a$. Then there are $p, q \in \mathbb{K}[x] \setminus \{0\}$ and $N \in \mathbb{N}$ s.t. $q(n)a_{n+1} + p(n)a_n = 0$ for all $n \in \mathbb{N}$, and $a'_n = a_n \neq 0$ for all $n \geq N$. Hence

$$n(n-1)\cdots(n-N+1)q(n)a'_{n+1} + n(n-1)\cdots(n-N+1)p(n)a'_n = 0$$

for all $n \in \mathbb{N}$, so $a' \in \mathcal{H}(\mathbb{K})$. □

Proposition 2. *Let $\mathcal{C} \subseteq \mathbb{K}^{\mathbb{N}}$ be a set of sequences closed under all inverse shifts and addition, and such that $0 \in \mathcal{C}$. Then $\mathcal{C}$ is closed under equivalence.*

Proof: Let $a \in \mathcal{C}$ and $a' \sim a$. Then there are $k \in \mathbb{N}$ and $\lambda_0, \lambda_1, \ldots, \lambda_k \in \mathbb{K}$ s.t.

$$a' - a = \langle \lambda_0, \lambda_1, \ldots, \lambda_k, 0, 0, 0, \ldots \rangle = E_{\lambda_0}^{-1} E_{\lambda_1}^{-1} \cdots E_{\lambda_k}^{-1}(0),$$

so $a' = a + E_{\lambda_0}^{-1} E_{\lambda_1}^{-1} \cdots E_{\lambda_k}^{-1}(0) \in \mathcal{C}$. □

Corollary 1. *The rings $\mathcal{A}(\mathbb{K})$, $\mathcal{L}(\mathbb{K})$, $\mathcal{A}_{rat}(\mathbb{K})$, $\mathcal{L}_{rat}(\mathbb{K})$, $\mathcal{A}_{qrat}(\mathbb{K})$, $\mathcal{L}_{qrat}(\mathbb{K})$, $\mathcal{P}(\mathbb{K})$ are closed under equivalence.*

Proposition 3. *While shift, inverse shift, difference, multisection, scalar multiplication, addition, multiplication, and interlacing are local operations, indefinite summation and convolution are not local.*

Proof: Straightforward. □

When dealing with local operations, it is customary to work with germs of sequences which simplifies the statements of results and their corresponding proofs. Since here we are especially interested in the nonlocal operation of convolution, we need to work with sequences themselves. In this situation, the following auxiliary results will prove to be useful.

Lemma 1. *Let $a, b, \varepsilon, \eta \in \mathbb{K}^{\mathbb{N}}$ with $\varepsilon, \eta \sim 0$. Then:*

(i) $a\,\varepsilon \sim 0$,

(ii) $\sum_{k=0}^{n} \varepsilon_k \sim C$ *for some* $C \in \mathbb{K}$,

(iii) $a * \varepsilon = \sum_{k=0}^{N} \varepsilon_k E_0^{-k}(a)$ *for some* $N \in \mathbb{N}$,

(iv) $\varepsilon * \eta \sim 0$,

(v) $(a+\eta)*(b+\varepsilon) \ \sim\ a*b\ + \sum_{i=0}^{N_1} \varepsilon_i E_0^{-i}(a)\ + \sum_{j=0}^{N_2} \eta_j E_0^{-j}(b)$ *for some* $N_1, N_2 \in \mathbb{N}$.

Proof:

(i) This follows from locality of multiplication.

(ii) Let $N \in \mathbb{N}$ be such that $\varepsilon_k = 0$ for $k > N$. Write $C = \sum_{k=0}^{N} \varepsilon_k$. For $n \geq N$ we have $\sum_{k=0}^{n} \varepsilon_k = \sum_{k=0}^{N} \varepsilon_k$, so $\sum_{k=0}^{n} \varepsilon_k \sim C$.

(iii) Let $N \in \mathbb{N}$ be such that $\varepsilon_k = 0$ for $k > N$. Then for all $n \in \mathbb{N}$,

$$(a*\varepsilon)_n = \sum_{k=0}^{n} \varepsilon_k a_{n-k} = \sum_{k=0}^{\min\{n,N\}} \varepsilon_k E_0^{-k}(a)_n = \left(\sum_{k=0}^{N} \varepsilon_k E_0^{-k}(a)\right)_n \tag{4}$$

where the last equality follows from the fact that $E_0^{-k}(a)_n = 0$ for $k > n$.

(iv) Let $N_1, N_2 \in \mathbb{N}$ be such that $\varepsilon_i = 0$ for $i > N_1$ and $\eta_j = 0$ for $j > N_2$. Assume that $n > N_1 + N_2$. Then $k > N_1$ or $n - k > N_2$ for every $k \in \mathbb{N}$, therefore

$$(\varepsilon * \eta)_n = \sum_{k=0}^{n} \varepsilon_k \eta_{n-k} = 0$$

for all such n, so $\varepsilon * \eta \sim 0$.

(v) By bilinearity and commutativity of convolution we have

$$(a+\eta)*(b+\varepsilon) = a*b + a*\varepsilon + b*\eta + \varepsilon*\eta.$$

The claim now follows from (iii) and (iv). □

5. Proof of Theorem 1 (the d'Alembertian case)

We use the fact that d'Alembertian sequences are annihilated by completely factorable linear recurrence operators with polynomial coefficients. This implies that each d'Alembertian sequence can be written as a linear combination of nested indefinite sums with hypergeometric factors in the summands. Similarly, each (quasi)-rationally d'Alembertian sequence can be written as a linear combination of nested indefinite sums with (quasi)-rational factors in the summands.

Definition 8. *For $d \in \mathbb{N} \setminus \{0\}$ and $a^{(1)}, a^{(2)}, \ldots, a^{(d)} \in \mathbb{K}^{\mathbb{N}}$, we shall denote by*

$$\mathrm{NS}\left(a^{(1)}, a^{(2)}, \ldots, a^{(d)}\right) \;=\; \mathrm{NS}_{i=1}^{d} a^{(i)}$$

the sequence $a \in \mathbb{K}^{\mathbb{N}}$ defined for all $k_1 \in \mathbb{N}$ by

$$a_{k_1} \;:=\; \left(\mathrm{NS}_{i=1}^{d} a^{(i)}\right)_{k_1} \;=\; a_{k_1}^{(1)} \sum_{k_2=0}^{k_1} a_{k_2}^{(2)} \sum_{k_3=0}^{k_2} a_{k_3}^{(3)} \cdots \sum_{k_d=0}^{k_{d-1}} a_{k_d}^{(d)} \tag{5}$$

and call it the nested sum *of sequences $a^{(1)}, a^{(2)}, \ldots, a^{(d)}$. Note that the scope of each summation sign on the right of* (5) *extends to the end of the formula. We will call the number d the* nesting depth *of this representation of a.*

Theorem 3. *Let $a \in \mathbb{K}^{\mathbb{N}}$. Then:*

(i) *a is d'Alembertian iff it can be written as a $\mathbb{K}$-linear combination (possibly empty) of nested sums of the form (5) where $a^{(1)}, a^{(2)}, \ldots, a^{(d)} \in \mathcal{H}(\mathbb{K})$,*

(ii) *a is d'Alembertian iff there are $d \in \mathbb{N} \setminus \{0\}$ and $L_1, L_2, \ldots, L_d \in \mathbb{K}[n]\langle E\rangle$, each of order 1, such that $L_1 L_2 \cdots L_d(a) = 0$.*

For a proof, see [3] or [10].

Corollary 2. *If $y \in \mathbb{K}^{\mathbb{N}}$ satisfies $L(y) = a$ where L is a product of first-order operators and $a \in \mathcal{A}(\mathbb{K})$, then $y \in \mathcal{A}(\mathbb{K})$.*

Proof: By Theorem 3.(ii), there are $d \in \mathbb{N} \setminus \{0\}$ and $L_1, L_2, \ldots, L_d \in \mathbb{K}[n]\langle E\rangle$, each of order 1, such that $L_1 L_2 \cdots L_d(a) = 0$. Hence

$$L_1 L_2 \cdots L_d L(y) \;=\; L_1 L_2 \cdots L_d(a) \;=\; 0,$$

so, again by Theorem 3.(ii), $y \in \mathcal{A}(\mathbb{K})$. $\square$

Theorem 4. *A sequence $a \in \mathbb{K}^{\mathbb{N}}$ is (quasi)-rationally d'Alembertian iff it can be written as a $\mathbb{K}$-linear combination (possibly empty) of nested sums of the form (5) where $a^{(1)}, a^{(2)}, \ldots, a^{(d)}$ are (quasi)-rational sequences.*

The proof is analogous to that of Theorem 3(i).

Proposition 4. *For all $k \in \mathbb{N}$ and $a, b \in \mathcal{A}(\mathbb{K})$, we have*

$$a * b \in \mathcal{A}(\mathbb{K}) \iff E^k(a) * E^k(b) \in \mathcal{A}(\mathbb{K}). \tag{6}$$

Proof: Note that, for all $n \in \mathbb{N}$,

$$\begin{aligned} E^2(a*b)_n &= \sum_{k=0}^{n+2} a_k b_{n+2-k} = a_0 b_{n+2} + a_{n+2} b_0 + \sum_{k=1}^{n+1} a_k b_{n+2-k} \\ &= a_0 b_{n+2} + a_{n+2} b_0 + \sum_{k=0}^{n} a_{k+1} b_{n+1-k} \\ &= a_0 E^2(b)_n + b_0 E^2(a)_n + (E(a) * E(b))_n. \end{aligned}$$

As $\mathcal{A}(\mathbb{K})$ is closed under shift, scalar multiplication and addition, this implies

$$E^2(a*b) \in \mathcal{A}(\mathbb{K}) \iff E(a) * E(b) \in \mathcal{A}(\mathbb{K}).$$

By the closure of $\mathcal{A}(\mathbb{K})$ under shift and all inverse shifts, we have

$$a * b \in \mathcal{A}(\mathbb{K}) \iff E^2(a*b) \in \mathcal{A}(\mathbb{K}),$$

so

$$a * b \in \mathcal{A}(\mathbb{K}) \iff E(a) * E(b) \in \mathcal{A}(\mathbb{K}). \tag{7}$$

As $\mathcal{A}(\mathbb{K})$ is closed under shift, we can replace a by $E^k(a)$ and b by $E^k(b)$ in (7) and obtain

$$E^k(a) * E^k(b) \in \mathcal{A}(\mathbb{K}) \iff E^{k+1}(a) * E^{k+1}(b) \in \mathcal{A}(\mathbb{K})$$

for all $k \in \mathbb{N}$. Now (6) follows by induction on k. $\square$

Lemma 2. *Let $d \in \mathbb{N}$, $a^{(1)}, a^{(2)}, \ldots, a^{(d)} \in \mathbb{K}^{\mathbb{N}}$, $\varepsilon^{(1)}, \varepsilon^{(2)}, \ldots, \varepsilon^{(d)} \in \mathbb{K}^{\mathbb{N}}$, and $\varepsilon^{(1)}, \varepsilon^{(2)}, \ldots, \varepsilon^{(d)} \sim 0$. Then there are $c_1, c_2, \ldots, c_d \in \mathbb{K}$ such that*

$$\mathrm{NS}_{i=1}^{d} \left(a^{(i)} + \varepsilon^{(i)}\right) \sim \sum_{i=1}^{d} c_i \, \mathrm{NS}_{j=1}^{i} a^{(j)}.$$

Proof: By induction on d.

If $d = 0$ both sides are 0. Now assume that the assertion holds at some $d \geq 1$, and expand the left-hand side. In line 2 we use the induction hypothesis and compensate for replacing equivalence with equality by adding a sequence $\eta \sim 0$ in the appropriate place. In lines 3 and 4 we use Lemma 1.(i) resp. (ii).

We denote the constant C introduced in Lemma 1.(ii) by c_1:

$$\begin{aligned}
\left(\mathrm{NS}_{i=1}^{d+1}\left(a^{(i)}+\varepsilon^{(i)}\right)\right)_n &= \left(a_n^{(1)}+\varepsilon_n^{(1)}\right)\sum_{k_2=0}^{n}\left(\mathrm{NS}_{i=2}^{d+1}\left(a^{(i)}+\varepsilon^{(i)}\right)\right)_{k_2}\\
&= \left(a_n^{(1)}+\varepsilon_n^{(1)}\right)\sum_{k_2=0}^{n}\left(\sum_{i=2}^{d+1}c_i\left(\mathrm{NS}_{j=2}^{i}\,a^{(j)}\right)_{k_2}+\eta_{k_2}\right)\\
&\sim a_n^{(1)}\sum_{k_2=0}^{n}\sum_{i=2}^{d+1}c_i\left(\mathrm{NS}_{j=2}^{i}\,a^{(j)}\right)_{k_2}+a_n^{(1)}\sum_{k_2=0}^{n}\eta_{k_2}\\
&\sim \sum_{i=2}^{d+1}c_i\left(\mathrm{NS}_{j=1}^{i}\,a^{(j)}\right)_n+c_1a_n^{(1)}\\
&= \sum_{i=1}^{d+1}c_i\left(\mathrm{NS}_{j=1}^{i}\,a^{(j)}\right)_n.
\end{aligned}$$

□

Lemma 3. *Let $d\in\mathbb{N}$ and $a^{(1)},a^{(2)},\ldots,a^{(d)}\in\mathbb{K}^{\mathbb{N}}$. If $N\in\mathbb{N}$ is s.t. $a_n^{(i)}=0$ for all $n<N$ and $i\in\{1,2,\ldots,d\}$, then*

$$E^N\left(\mathrm{NS}_{i=1}^{d}a^{(i)}\right) \;=\; \mathrm{NS}_{i=1}^{d}\,E^N\left(a^{(i)}\right).$$

Proof: Write the nested sum on the left as a single sum, shift all summation indices by N, and use the fact that all original summands vanish below N:

$$\begin{aligned}
\left(E^N\left(\mathrm{NS}_{i=1}^{d}a^{(i)}\right)\right)_n &= a_{n+N}^{(1)}\sum_{k_2=0}^{n+N}a_{k_2}^{(2)}\sum_{k_3=0}^{k_2}a_{k_3}^{(3)}\cdots\sum_{k_d=0}^{k_{d-1}}a_{k_d}^{(d)}\\
&= \sum_{0\le k_d\le\cdots\le k_3\le k_2\le n+N}a_{n+N}^{(1)}a_{k_2}^{(2)}a_{k_3}^{(3)}\cdots a_{k_d}^{(d)}\\
&= \sum_{-N\le k_d\le\cdots\le k_3\le k_2\le n}a_{n+N}^{(1)}a_{k_2+N}^{(2)}a_{k_3+N}^{(3)}\cdots a_{k_d+N}^{(d)}\\
&= \sum_{0\le k_d\le\cdots\le k_3\le k_2\le n}a_{n+N}^{(1)}a_{k_2+N}^{(2)}a_{k_3+N}^{(3)}\cdots a_{k_d+N}^{(d)}\\
&= \left(\mathrm{NS}_{i=1}^{d}\,E^N\left(a^{(i)}\right)\right)_n.
\end{aligned}$$

□

In Proposition 5 we prove the "ideal" case of Theorem 1 where the minimal annihilators of the hypergeometric resp. rational sequences in the two factors are nonsingular. We use induction on the sum of nesting depths of both factors and of the valuation of the quasi-rational factor. Thanks to the Partial Fraction Decomposition Theorem for rational functions, it suffices to consider three cases according to the possible forms of a term in the partial fraction decomposition of a quasi-rational function, namely α^x, $\alpha^x x^j$, and $\frac{\alpha^x}{(x-\beta)^j}$ (both with $j \geq 1$). In each of the cases, we apply to the convolution $a * b$ a linear recurrence operator L_0 of order at most 1 such that the nesting depth of one of the factors or the valuation of the quasi-rational factor decreases. Then we use the inductive hypothesis to show that $L_0(a * b)$ is d'Alembertian, and invoke Corollary 2 on p. 8 to conclude that so is $a * b$.

Proposition 5. *Let $d, e \in \mathbb{N}$, and let for all $n \in \mathbb{N}$*

$$a_n = \begin{cases} 0 & \text{if } d = 0 \\ h_n^{(1)} \sum_{k_2=0}^{n+\eta_1} h_{k_2}^{(2)} \sum_{k_3=0}^{k_2+\eta_2} h_{k_3}^{(3)} \cdots \sum_{k_d=0}^{k_{d-1}+\eta_{d-1}} h_{k_d}^{(d)} & \text{if } d \geq 1, \end{cases}$$

$$b_n = \begin{cases} 0 & \text{if } e = 0 \\ \varphi_1(n) \sum_{k_2=0}^{n+\xi_1} \varphi_2(k_2) \sum_{k_3=0}^{k_2+\xi_2} \varphi_3(k_3) \cdots \sum_{k_e=0}^{k_{e-1}+\xi_{e-1}} \varphi_e(k_e) & \text{if } e \geq 1 \end{cases}$$

where

- *for $i = 1, 2, \ldots, d$, $h^{(i)}$ is hypergeometric over $\mathbb{K}$, and there are $p_i, q_i \in \mathbb{K}[x]$ such that $q_i(n) \neq 0$ and $q_i(n)\ h_{n+1}^{(i)} = p_i(n)\ h_n^{(i)}$ for all $n \in \mathbb{N}$,*
- *$\eta_1, \eta_2, \ldots, \eta_{d-1} \in \mathbb{N}$,*
- *for $i = 1, 2, \ldots, e$, there are $\alpha_i \in \mathbb{K}^*$, $\beta_i \in \mathbb{K} \setminus \mathbb{N}$ and $j_i \in \mathbb{N}$ such that $\varphi_i(x) \in \{\alpha_i^x x^{j_i}, \alpha_i^x (x - \beta_i)^{-j_i}\}$,*
- *$\xi_1, \xi_2, \ldots, \xi_{e-1} \in \mathbb{N}$.*

Then $a * b$ is d'Alembertian.

Proof: Note that a is d'Alembertian, and b is quasi-rationally d'Alembertian. We use induction on $d + e + \sum_{i=1}^e j_i$ where d resp. e are the nesting depths of these representations of a resp. b, and $\sum_{i=1}^e j_i$ is the *valuation* of b. If $d = 0$ or $e = 0$, then $a * b = 0 \in \mathcal{A}(\mathbb{K})$. Now let $d, e \geq 1$. Write

$$a_n = h_n \sum_{k_2=0}^{n+\eta} \tilde{a}_{k_2} \tag{8}$$

where $h = h^{(1)}$, $\eta = \eta_1$ and

$$\tilde{a}_{k_2} = \begin{cases} \delta_{k_2,0}, & d = 1, \\ h_{k_2}^{(2)} \sum_{k_3=0}^{k_2+\eta_2} h_{k_3}^{(3)} \cdots \sum_{k_d=0}^{k_{d-1}+\eta_{d-1}} h_{k_d}^{(d)}, & d \geq 2 \end{cases}$$

with $\delta_{k_2,0} = \langle 1, 0, 0, 0, \ldots \rangle \in \mathbb{K}^{\mathbb{N}}$ the identity element for convolution. Write

$$b_n = \varphi(n) \sum_{k_2=0}^{n+\xi} \tilde{b}_{k_2} \tag{9}$$

where $\varphi = \varphi_1$ (with $j = j_1$, $\alpha = \alpha_1$, $\beta = \beta_1$), $\xi = \xi_1$ and

$$\tilde{b}_{k_2} = \begin{cases} \delta_{k_2,0}, & e = 1, \\ \varphi_2(k_2) \displaystyle\sum_{k_3=0}^{k_2+\xi_2} \varphi_3(k_3) \cdots \sum_{k_e=0}^{k_{e-1}+\xi_{e-1}} \varphi_e(k_e), & e \geq 2. \end{cases}$$

We shall prove that the convolution

$$y_n := \sum_{k=0}^{n} a_k b_{n-k} = \sum_{k=0}^{n} h_k \varphi(n-k) \left(\sum_{k_2=0}^{k+\eta} \tilde{a}_{k_2} \right) \left(\sum_{k_2=0}^{n-k+\xi} \tilde{b}_{k_2} \right)$$

is d'Alembertian by showing that $L_0(y) \in \mathcal{A}(\mathbb{K})$ for an appropriate operator $L_0 \in \mathbb{K}[n]\langle E \rangle$, then invoking Corollary 2. We distinguish three cases:

Case 1. $\varphi(x) = \alpha^x$

In this case $y_n = \sum_{k=0}^{n} a_k \alpha^{n-k} \sum_{k_2=0}^{n-k+\xi} \tilde{b}_{k_2}$ and we take $L_0 = E - \alpha$. Then

$$\begin{aligned} (L_0(y))_n &= y_{n+1} - \alpha y_n \\ &= \sum_{k=0}^{n+1} a_k \alpha^{n+1-k} \sum_{k_2=0}^{n+1-k+\xi} \tilde{b}_{k_2} - \sum_{k=0}^{n} a_k \alpha^{n+1-k} \sum_{k_2=0}^{n-k+\xi} \tilde{b}_{k_2} \\ &= a_{n+1} \sum_{k_2=0}^{\xi} \tilde{b}_{k_2} + \sum_{k=0}^{n} a_k \alpha^{n+1-k} \tilde{b}_{n-k+\xi+1} \\ &= E(a)_n \sum_{k_2=0}^{\xi} \tilde{b}_{k_2} + \alpha \left(a_n * \alpha^n E^{\xi+1} \tilde{b}_n \right) \end{aligned}$$

where $E(a)$ is d'Alembertian and $\alpha^n E^{\xi+1} \tilde{b}_n$ has nesting depth $e - 1$, hence $a_n * \alpha^n E^{\xi+1} \tilde{b}_n$ is d'Alembertian by induction hypothesis.

Case 2. $\varphi(x) = \alpha^x x^j$ with $j \geq 1$

Here $b_n = n^j c_n$ where $c_n = \alpha^n \sum_{k_2=0}^{n+\xi} \tilde{b}_{k_2}$, and we take $L_0 = 1$. So

$$\begin{aligned} y_n &= \sum_{k=0}^{n} a_k (n-k)^j c_{n-k} = \sum_{i=0}^{j} (-1)^i \binom{j}{i} n^{j-i} \sum_{k=0}^{n} k^i a_k c_{n-k} \\ &= \sum_{i=0}^{j} (-1)^i \binom{j}{i} n^{j-i} \left((n^i a_n) * c_n \right). \end{aligned}$$

As the valuation of c is j less than that of b, our induction hypothesis implies that y is d'Alembertian.

Case 3. $\varphi(x) = \frac{\alpha^x}{(x-\beta)^j}$ with $j \geq 1$

Here we take $L_0 = q(n-\beta)E - p(n-\beta)$ where polynomials $p, q \in \mathbb{K}[x] \setminus \{0\}$ are such that $q(n)h_{n+1} - p(n)h_n = 0$ for all $n \in \mathbb{N}$. Then

$$\begin{aligned}
(L_0(y))_n &= \\
&= q(n-\beta)\sum_{k=0}^{n+1} \frac{h_k \alpha^{n+1-k}}{(n+1-k-\beta)^j} \left(\sum_{k_2=0}^{k+\eta} \tilde{a}_{k_2}\right) \left(\sum_{k_2=0}^{n+1-k+\xi} \tilde{b}_{k_2}\right) \\
&- p(n-\beta)\sum_{k=0}^{n} \frac{h_k \alpha^{n-k}}{(n-k-\beta)^j} \left(\sum_{k_2=0}^{k+\eta} \tilde{a}_{k_2}\right) \left(\sum_{k_2=0}^{n-k+\xi} \tilde{b}_{k_2}\right) \\
&= q(n-\beta)\sum_{k=-1}^{n} \frac{h_{k+1} \alpha^{n-k}}{(n-k-\beta)^j} \left(\sum_{k_2=0}^{k+1+\eta} \tilde{a}_{k_2}\right) \left(\sum_{k_2=0}^{n-k+\xi} \tilde{b}_{k_2}\right) \\
&- p(n-\beta)\sum_{k=0}^{n} \frac{h_k \alpha^{n-k}}{(n-k-\beta)^j} \left(\sum_{k_2=0}^{k+\eta} \tilde{a}_{k_2}\right) \left(\sum_{k_2=0}^{n-k+\xi} \tilde{b}_{k_2}\right) \\
&= A_n + B_n + C_n,
\end{aligned}$$

where

$$\begin{aligned}
A_n &:= q(n-\beta)a_0\, b_{n+1}, \\
B_n &:= q(n-\beta)\sum_{k=0}^{n} h_{k+1}\tilde{a}_{k+1+\eta} \frac{\alpha^{n-k}}{(n-k-\beta)^j} \left(\sum_{k_2=0}^{n-k+\xi} \tilde{b}_{k_2}\right), \\
C_n &:= \sum_{k=0}^{n} \frac{q(n-\beta)h_{k+1} - p(n-\beta)h_k}{(n-k-\beta)^j} \alpha^{n-k} \left(\sum_{k_2=0}^{k+\eta} \tilde{a}_{k_2}\right) \left(\sum_{k_2=0}^{n-k+\xi} \tilde{b}_{k_2}\right).
\end{aligned}$$

Clearly A is d'Alembertian. Since $B_n = q(n-\beta)\left(E(hE^\eta(\tilde{a})) * b\right)_n$ and the nesting depth of $E(hE^\eta(\tilde{a}))$ is $d-1$, B is d'Alembertian by the induction hypothesis. In C_n we replace h_{k+1} with $h_k\, p(k)/q(k)$ and obtain

$$C_n = \sum_{k=0}^{n} \frac{P(k)h_k \alpha^{n-k}}{q(k)(n-k-\beta)^j} \left(\sum_{k_2=0}^{k+\eta} \tilde{a}_{k_2}\right) \left(\sum_{k_2=0}^{n-k+\xi} \tilde{b}_{k_2}\right)$$

where $P(k) := q(n-\beta)p(k) - p(n-\beta)q(k) \in \mathbb{K}[n][k]$. Since $P(n-\beta) = 0$, $P(k)$ is divisible by $k - n + \beta$, hence there are $s \in \mathbb{N}$ and $c_0, c_1, \ldots, c_s \in \mathbb{K}[x]$ such that $P(k) = (n-k-\beta)\sum_{i=0}^{s} c_i(n)k^i$. It follows that

$$\begin{aligned}
C_n &= \sum_{i=0}^{s} c_i(n) \sum_{k=0}^{n} \frac{k^i a_k}{q(k)} \cdot \frac{\alpha^{n-k}}{(n-k-\beta)^{j-1}} \left(\sum_{k_2=0}^{n-k+\xi} \tilde{b}_{k_2}\right) \\
&= \sum_{i=0}^{s} c_i(n) \sum_{k=0}^{n} u_k^{(i)} \cdot \frac{\alpha^{n-k}}{(n-k-\beta)^{j-1}} \left(\sum_{k_2=0}^{n-k+\xi} \tilde{b}_{k_2}\right) \\
&= \sum_{i=0}^{s} c_i(n) \left(u_n^{(i)} * (n-\beta)b_n\right)
\end{aligned}$$

where $u_k^{(i)} := \frac{k^i a_k}{q(k)}$ for all $k \in \mathbb{N}$ and $i \in \{1, 2, \ldots, s\}$. As the valuation of $(n-\beta)b_n$ is one less than that of b_n, our induction hypothesis implies that $u_n^{(i)} * (n-\beta)b_n$ is d'Alembertian, hence so are C and $L_0(y) = A + B + C$. Since L_0 has order one, Corollary 2 implies that $y = a * b$ is d'Alembertian. $\square$

Based on Proposition 5, we now prove the full version of Theorem 1. Again we use induction on the nesting depth of the two factors, as well as Lemmas 2, 3 and Proposition 4.

Proof of **Theorem 1** (see p. 5): By Theorem 3.(i), the sequence a can be written as a $\mathbb{K}$-linear combination of sequences of the form $\mathrm{NS}_{i=1}^{d} h^{(i)}$ where $h^{(1)}, h^{(2)}, \ldots, h^{(d)} \in \mathcal{H}(\mathbb{K})$. For $i = 1, 2, \ldots, d$, let $p_i, q_i \in \mathbb{K}[x]$ be such that $q_i(n) h_{n+1}^{(i)} = p_i(n) h_n^{(i)}$ for all $n \in \mathbb{N}$. By Theorem 4, the sequence b can be written as a $\mathbb{K}$-linear combination of sequences of the form $\mathrm{NS}_{i=1}^{e} r^{(i)}$ where $r^{(1)}, r^{(2)}, \ldots, r^{(e)}$ are (quasi)-rational sequences. By the Partial Fraction Decomposition Theorem for rational functions, we can assume that for $i = 1, 2, \ldots, e$ there are $j_i \in \mathbb{N}$, $\alpha_i \in \mathbb{K}^*$ and $\beta_i \in \mathbb{K}$ such that $r_n^{(i)} = \varphi_i(n)$ for all large enough n, where $\varphi_i(x) \in \{\alpha_i^x x^{j_i}, \alpha_i^x (x - \beta_i)^{-j_i}\}$.

By bilinearity of convolution, it suffices to prove that the convolution of a single $\mathrm{NS}_{i=1}^{d} h^{(i)}$ with a single $\mathrm{NS}_{i=1}^{e} r^{(i)}$ is d'Alembertian, so henceforth we assume that $a \equiv \mathrm{NS}_{i=1}^{d} h^{(i)}$ and $b \equiv \mathrm{NS}_{i=1}^{e} r^{(i)}$. Let $N \in \mathbb{N}$ be such that for all $n \geq N$, $q_i(n) \neq 0$ for all $i \in \{1, 2, \ldots, d\}$, and $r_n^{(i)} = \varphi_i(n)$ for all $i \in \{1, 2, \ldots, e\}$. We shall prove by induction on the sum of nesting depths $d + e$ that $a * b$ is d'Alembertian.

If $d = 0$ or $e = 0$ then $a = 0$ or $b = 0$ and so $a * b = 0 \in \mathcal{A}(\mathbb{K})$.

Assume now that $d \geq 1$ and $e \geq 1$. Let $\tilde{a} := \mathrm{NS}_{k=1}^{d} \tilde{h}^{(k)}$ and $\tilde{b} := \mathrm{NS}_{k=1}^{e} \tilde{r}^{(k)}$ where $\tilde{h}^{(k)} = E_0^{-N} E^N \left(h^{(k)}\right)$ and $\tilde{r}^{(k)} = E_0^{-N} E^N \left(r^{(k)}\right)$. Then $\tilde{h}_n^{(k)} = \tilde{r}_n^{(k)} = 0$ for $n < N$ and $\tilde{h}_n^{(k)} = h_n^{(k)}$, $\tilde{r}_n^{(k)} = r_n^{(k)}$ for $n \geq N$. It follows by Lemma 3 that

$$E^N(\tilde{a})_n = \mathrm{NS}_{k=1}^{d} E^N \left(\tilde{h}^{(k)}\right)_n = \mathrm{NS}_{k=1}^{d} E^N \left(h^{(k)}\right)_n = \mathrm{NS}_{k=1}^{d} h_{n+N}^{(k)},$$

$$E^N(\tilde{b})_n = \mathrm{NS}_{k=1}^{e} E^N \left(\tilde{r}^{(k)}\right)_n = \mathrm{NS}_{k=1}^{e} E^N \left(r^{(k)}\right)_n = \mathrm{NS}_{k=1}^{e} \varphi_k(n+N).$$

Note that by our definition of N, the sequences $E^N(\tilde{a})$ and $E^N(\tilde{b})$ satisfy all the assumptions of Proposition 5, so $E^N(\tilde{a}) * E^N(\tilde{b}) \in \mathcal{A}(\mathbb{K})$. Proposition 4 now implies that $\tilde{a} * \tilde{b} \in \mathcal{A}(\mathbb{K})$ as well.

By Lemma 2, there are $c_1, c_2, \ldots, c_d \in \mathbb{K}$ and $c'_1, c'_2, \ldots, c'_e \in \mathbb{K}$ such that

$$a = \sum_{i=1}^{d} c_i \, \mathrm{NS}_{j=1}^{i} \tilde{h}^{(j)} + \eta = c_d \, \mathrm{NS}_{j=1}^{d} \tilde{h}^{(j)} + \sum_{i=1}^{d-1} c_i \, \mathrm{NS}_{j=1}^{i} \tilde{h}^{(j)} + \eta,$$

$$b = \sum_{i=1}^{e} c'_i \, \mathrm{NS}_{j=1}^{i} \tilde{r}^{(j)} + \eta' = c'_e \, \mathrm{NS}_{j=1}^{e} \tilde{r}^{(j)} + \sum_{i=1}^{e-1} c'_i \, \mathrm{NS}_{j=1}^{i} \tilde{r}^{(j)} + \eta'$$

for some sequences $\eta, \eta' \sim 0$. Hence

$$\begin{aligned}
a * b \;&=\; c_d\, c'_e\, \mathrm{NS}_{j=1}^{d} \tilde{h}^{(j)} * \mathrm{NS}_{j=1}^{e} \tilde{r}^{(j)} + c_d\, \mathrm{NS}_{j=1}^{d} \tilde{h}^{(j)} * \sum_{i=1}^{e-1} c'_i\, \mathrm{NS}_{j=1}^{i} \tilde{r}^{(j)} \\
&+\; c'_e\, \mathrm{NS}_{j=1}^{e} \tilde{r}^{(j)} * \sum_{i=1}^{d-1} c_i\, \mathrm{NS}_{j=1}^{i} \tilde{h}^{(j)} + \sum_{i=1}^{d-1} c_i\, \mathrm{NS}_{j=1}^{i} \tilde{h}^{(j)} * \sum_{i=1}^{e-1} c'_i\, \mathrm{NS}_{j=1}^{i} \tilde{r}^{(j)} \\
&+\; \eta * \left(c'_e\, \mathrm{NS}_{j=1}^{e} \tilde{r}^{(j)} + \sum_{i=1}^{e-1} c'_i\, \mathrm{NS}_{j=1}^{i} \tilde{r}^{(j)} + \eta' \right) \\
&+\; \eta' * \left(c_d\, \mathrm{NS}_{j=1}^{d} \tilde{h}^{(j)} + \sum_{i=1}^{d-1} c_i\, \mathrm{NS}_{j=1}^{i} \tilde{h}^{(j)} \right).
\end{aligned}$$

The first term on the right equals $c_d\, c'_e\, \tilde{a} * \tilde{b}$, so it is d'Alembertian as shown in the preceding paragraph. The next three terms are linear combinations of convolutions of nested sums having nesting depths at most $d+e-1$, $d+e-1$, and $d+e-2$, respectively, so they are d'Alembertian by induction hypothesis. By Lemma 1.(iii), the last two terms above are linear combinations of shifted d'Alembertian sequences, so they are d'Alembertian as well. It follows that $a*b$ is d'Alembertian as claimed. $\square$

Example 6. *By Theorem 1, the convolution of a hypergeometric sequence with a rational sequence, such as*

$$y_n \;=\; (2^{n-1} n!) * \left(\frac{1}{n+\frac{1}{2}} \right) \;=\; \sum_{k=0}^{n} \frac{2^{k-1} k!}{n-k+\frac{1}{2}},$$

is d'Alembertian. By following through the proof of Proposition 5 with $a_n = 2^{n-1} n!$ and $b_n = 1/(n+\frac{1}{2})$, we will obtain an explicit nested-sum representation of y_n. Here the nesting depths of a and b are 1, $j = 1$, $\beta = -1/2$, $h_n = a_n = 2^{n-1} n!$, $h_{n+1}/h_n = p(n) = 2(n+1)$, $q(n) = 1$ and

$$L_0 \;=\; q(n-\beta)E - p(n-\beta) \;=\; E - (2n+3).$$

Applying L_0 to $y(n)$ we obtain

$$
\begin{aligned}
(L_0(y))_n &= \sum_{k=0}^{n+1} \frac{2^{k-1}k!}{n-k+\frac{3}{2}} - (2n+3)\sum_{k=0}^{n} \frac{2^{k-1}k!}{n-k+\frac{1}{2}} \\
&= \sum_{k=-1}^{n} \frac{2^{k}(k+1)!}{n-k+\frac{1}{2}} - (2n+3)\sum_{k=0}^{n} \frac{2^{k-1}k!}{n-k+\frac{1}{2}} \\
&= \frac{1}{2n+3} + \sum_{k=0}^{n} \frac{2^{k-1}k!(2(k+1)-(2n+3))}{n-k+\frac{1}{2}} \\
&= \frac{1}{2n+3} + \sum_{k=0}^{n} \frac{2^{k-1}k!(2k-2n-1)}{n-k+\frac{1}{2}} \\
&= \frac{1}{2n+3} - \sum_{k=0}^{n} 2^k k! \;=\; \frac{1}{2n+3} - \sum_{k=0}^{n} (2k)!!. \qquad (10)
\end{aligned}
$$

By solving this recurrence with initial condition $y_0 = 1$, we obtain

$$
y_n \;=\; (2n+1)!! \left(1 + \sum_{k=1}^{n} \frac{1}{(2k+1)!!}\left(\frac{1}{2k+1} - \sum_{j=0}^{k-1}(2j)!!\right)\right).
$$

Since $(2n+1)!!$ and $(2n)!!$ are hypergeometric sequences, this shows that $y(n)$ is indeed a d'Alembertian sequence.

From (10) *we can also obtain a fully factored annihilator of y as follows: The right-hand side of* (10) *is annihilated by the least common left multiple of $E - (2n+3)/(2n+5)$ and $(E-(2n+4))(E-1)$, which is*

$$
\left(E - \frac{(2n+3)(2n+7)^2}{(2n+5)^2(2n+9)}\right)(E-(2n+4))\,(E-1),
$$

hence $L(y) = 0$ where

$$
L \;=\; \left(E - \frac{(2n+3)(2n+7)^2}{(2n+5)^2(2n+9)}\right)(E-(2n+4))\,(E-1)(E-(2n+3)).
$$

6. Proof of Theorem 2 (the Liouvillian case)

To prove Theorem 2, we use generating functions, Theorem 1, and the fact that a sequence is Liouvillian if and only if it is an interlacing of d'Alembertian sequences [11, 10]. Recall that the (ordinary) generating function of a sequence $a \in \mathbb{K}^{\mathbb{N}}$ is defined as the formal power series

$$
g_a(x) = \sum_{k=0}^{\infty} a_k x^k,
$$

and that for all pairs of sequences $a, b \in \mathbb{K}^{\mathbb{N}}$ we have: $g_{a+b}(x) = g_a(x) + g_b(x)$, $g_{a*b}(x) = g_a(x)g_b(x)$.

Definition 9. [7] *For $m \in \mathbb{N}\setminus\{0\}$ and $a \in \mathbb{K}^{\mathbb{N}}$, we write $\Lambda^m a$ for $\Lambda(a, \overbrace{0, \ldots, 0}^{m-1})$, and call it the* 0^{th} m-interlacing of a with zeroes.

Lemma 4. *Let $k \in \mathbb{N}$, $m \in \mathbb{N} \setminus \{0\}$, $a, a^{(0)}, a^{(1)}, \ldots, a^{(m-1)} \in \mathbb{K}^{\mathbb{N}}$, and $b = \Lambda_{j=0}^{m-1} a^{(j)}$.*

(i) $\left(E_0^{-k}(a)\right)_n = \begin{cases} a_{n-k}, & n \geq k \\ 0, & n < k \end{cases}$

(ii) $(\Lambda^m a)_n = \begin{cases} a_{\frac{n}{m}}, & n \equiv 0 \pmod{m} \\ 0, & n \not\equiv 0 \pmod{m} \end{cases}$

(iii) $g_{E_0^{-k}(a)}(x) = x^k g_a(x)$

(iv) $g_{\Lambda^m a}(x) = g_a(x^m)$

(v) $g_b(x) = \sum_{j=0}^{m-1} x^j g_{a^{(j)}}(x^m)$

(vi) $\Lambda_{j=0}^{m-1} a^{(j)} = \sum_{j=0}^{m-1} E_0^{-j}\left(\Lambda^m a^{(j)}\right)$

(vii) $\Lambda^m E_0^{-k} = E_0^{-km}\Lambda^m$

Proof: Items (i), (ii) follow immediately from the definitions of E_0^{-1} and Λ^m.

(iii): $$g_{E_0^{-k}(a)}(x) = \sum_{n=0}^{\infty} E_0^{-k}(a)_n x^n = \sum_{n=k}^{\infty} a_{n-k} x^n = \sum_{n=0}^{\infty} a_n x^{n+k} = x^k g_a(x)$$

(iv): $$g_{\Lambda^m a}(x) = \sum_{n=0}^{\infty} (\Lambda^m a)_n x^n = \sum_{n \equiv 0 \pmod{m}} a_{\frac{n}{m}} x^n = \sum_{k=0}^{\infty} a_k x^{km} = g_a(x^m)$$

(v): $g_b(x) = \sum_{n=0}^{\infty} a_{n \operatorname{div} m}^{(n \bmod m)} x^n = \sum_{j=0}^{m-1} \sum_{k=0}^{\infty} a_k^{(j)} x^{km+j} = \sum_{j=0}^{m-1} x^j g_{a^{(j)}}(x^m)$

(vi): Using (v), (iv) and (iii) we find that

$$\begin{aligned} g_b(x) &= \sum_{j=0}^{m-1} x^j g_{a^{(j)}}(x^m) = \sum_{j=0}^{m-1} x^j g_{\Lambda^m a^{(j)}}(x) = \sum_{j=0}^{m-1} g_{E_0^{-j}\left(\Lambda^m a^{(j)}\right)}(x) \\ &= g_{\sum_{j=0}^{m-1} E_0^{-j}\left(\Lambda^m a^{(j)}\right)}(x) \end{aligned}$$

which implies the assertion.

(vii): By applying (iv) and (iii) alternatingly, we obtain

$$g_{\Lambda^m E_0^{-k}(a)}(x) = g_{E_0^{-k}(a)}(x^m) = x^{km} g_a(x^m) = x^{km} g_{\Lambda^m a}(x) = g_{E_0^{-km}\Lambda^m a}(x)$$

for every $a \in \mathbb{K}^{\mathbb{N}}$, which implies the assertion. □

Proposition 6. *The convolution of the* 0^{th} *m-interlacings of $a, b \in \mathbb{K}^{\mathbb{N}}$ with zeroes is the* 0^{th} *m-interlacing of $a * b$ with zeroes:*

$$\Lambda^m a * \Lambda^m b = \Lambda^m (a * b).$$

Proof:

$$\begin{aligned}
g_{\Lambda^m a * \Lambda^m b}(x) &= g_{\Lambda^m a}(x) g_{\Lambda^m b}(x) = g_a(x^m) g_b(x^m) \\
&= \sum_{i=0}^{\infty} a_i x^{mi} \sum_{j=0}^{\infty} b_j x^{mj} = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_i b_j x^{m(i+j)} \\
&= \sum_{k=0}^{\infty} x^{mk} \sum_{i=0}^{k} a_i b_{k-i} = \sum_{k=0}^{\infty} (a * b)_k \, (x^m)^k \\
&= g_{a*b}(x^m) = g_{\Lambda^m (a*b)}(x)
\end{aligned}$$

by using Lemma 4.(iv) three times. □

Proposition 7. *Let $m \in \mathbb{N} \setminus \{0\}$, $a^{(j)}, b^{(j)} \in \mathbb{K}^{\mathbb{N}}$ for all $j \in \{0, 1, \ldots, m-1\}$, $u = \Lambda_{j=0}^{m-1} a^{(j)}$, and $v = \Lambda_{j=0}^{m-1} b^{(j)}$. Then*

$$u * v = \sum_{k=0}^{2m-2} \sum_{j=\max\{0,k-m+1\}}^{\min\{k,m-1\}} E_0^{-k} \Lambda^m \left(a^{(j)} * b^{(k-j)} \right). \tag{11}$$

Proof: Using Lemma 4.(v), we obtain

$$\begin{aligned}
g_{u*v}(x) &= g_u(x) g_v(x) = \sum_{j=0}^{m-1} x^j g_{a^{(j)}}(x^m) \sum_{\ell=0}^{m-1} x^\ell g_{b^{(\ell)}}(x^m) \\
&= \sum_{j=0}^{m-1} \sum_{\ell=0}^{m-1} x^{j+\ell} g_{a^{(j)}}(x^m) g_{b^{(\ell)}}(x^m) \\
&= \sum_{k=0}^{m-1} \sum_{j=0}^{k} + \sum_{k=m}^{2m-2} \sum_{j=k-m+1}^{m-1} x^k g_{a^{(j)}}(x^m) g_{b^{(k-j)}}(x^m) \\
&= \sum_{k=0}^{2m-2} \sum_{j=\max\{0,k-m+1\}}^{\min\{k,m-1\}} x^k g_{a^{(j)}}(x^m) g_{b^{(k-j)}}(x^m) \qquad (12)
\end{aligned}$$

By Lemma 4.(iv), Proposition 6 and Lemma 4.(iii),

$$\begin{aligned}
x^k g_{a^{(j)}}(x^m) g_{b^{(k-j)}}(x^m) &= x^k g_{\Lambda^m a^{(j)}}(x) g_{\Lambda^m b^{(k-j)}}(x) = x^k g_{\Lambda^m a^{(j)} * \Lambda^m b^{(k-j)}}(x) \\
&= x^k g_{\Lambda^m \left(a^{(j)} * b^{(k-j)}\right)}(x) = g_{E_0^{-k} \Lambda^m \left(a^{(j)} * b^{(k-j)}\right)}(x)
\end{aligned}$$

which, together with (12), implies (11). □

Proof of **Theorem 2** (see p. 5): Let $u = \Lambda_{i=0}^{m-1} a^{(i)}$ with all $a^{(i)}$ d'Alembertian, and $v = \Lambda_{i=0}^{k-1} b^{(i)}$ with all $b^{(i)}$ (quasi)-rationally d'Alembertian. Let $\ell = \mathrm{lcm}(m, k)$. Write $u = \Lambda_{j=0}^{\ell-1} c^{(j)}$, $v = \Lambda_{j=0}^{\ell-1} d^{(j)}$ where $c^{(j)}$ and $d^{(j)}$, for $j = 0, 1, \ldots, \ell - 1$, are the j-th l-sections of u and v, respectively. Clearly all $c^{(j)}, d^{(j)}$ are themselves sections of $a^{(i)}$ resp. $b^{(i)}$. Since $\mathcal{A}(\mathbb{K})$ is closed under multisection [10, Prop. 7], all $c^{(j)}$ are d'Alembertian. Similarly one can show that the ring of (quasi)-rationally d'Alembertian sequences is closed under multisection, hence all $d^{(j)}$ are (quasi)-rationally d'Alembertian. So by Theorem 1, all convolutions $c^{(j_1)} * d^{(j_2)}$ for $j_1, j_2 \in \{0, 1, \dots, \ell-1\}$ are d'Alembertian. It follows from Proposition 7 that $u * v$ is a sum of shifted interlacings of d'Alembertian sequences, hence it is Liouvillian. □

Example 7. *By Theorem 2, the convolution of a Liouvillian sequence with a rational sequence, such as*

$$y_n \;:=\; n!! * \left(\frac{1}{n+1}\right) \;=\; \sum_{k=0}^{n} \frac{k!!}{n-k+1},$$

is Liouvillian. By following the proof of Proposition 7 with $u_n = n!!$ and $v_n = \frac{1}{n+1}$, we will obtain a representation of y_n as an interlacing of d'Alembertian sequences. Here $m = 2$, $u = \Lambda(a^{(0)}, a^{(1)})$ and $v = \Lambda(b^{(0)}, b^{(1)})$, where

$$\begin{aligned} a_n^{(0)} &= (2n)!! \;=\; 2^n n!, \\ a_n^{(1)} &= (2n+1)!! \;=\; \frac{(2n+1)!}{2^n n!}, \\ b_n^{(0)} &= v_{2n} \;=\; \frac{1}{2n+1}, \\ b_n^{(1)} &= v_{2n+1} \;=\; \frac{1}{2n+2}. \end{aligned}$$

By Proposition 7 at $m = 2$,

$$u * v \;=\; \Lambda^2\left(a^{(0)} * b^{(0)}\right) + E_0^{-1}\Lambda^2\left(a^{(0)} * b^{(1)} + a^{(1)} * b^{(0)}\right) + E_0^{-2}\Lambda^2\left(a^{(1)} * b^{(1)}\right).$$

Denote

$$\begin{aligned} g^{(0)} &:= a^{(0)} * b^{(0)} + E^{-1}\left(a^{(1)} * b^{(1)}\right), \\ g^{(1)} &:= a^{(0)} * b^{(1)} + a^{(1)} * b^{(0)}. \end{aligned}$$

For any $a, b, c, d \in \mathbb{K}^{\mathbb{N}}$ we have $\Lambda(a+b, c+d) = \Lambda(a, c) + \Lambda(b, d)$, therefore

$$\Lambda\left(g^{(0)}, g^{(1)}\right) = \Lambda\left(a^{(0)} * b^{(0)}, a^{(0)} * b^{(1)}\right) + \Lambda\left(E_0^{-1}\left(a^{(1)} * b^{(1)}\right), a^{(1)} * b^{(0)}\right),$$

which by Lemma 4.(vi) at $m = 2$ equals

$$\Lambda^2\left(a^{(0)} * b^{(0)}\right) + E_0^{-1}\Lambda^2\left(a^{(0)} * b^{(1)} + a^{(1)} * b^{(0)}\right) + \Lambda^2 E_0^{-1}\left(a^{(1)} * b^{(1)}\right).$$

Since $\Lambda^2 E_0^{-1}\left(a^{(1)} * b^{(1)}\right) = E_0^{-2}\Lambda^2\left(a^{(1)} * b^{(1)}\right)$ by Lemma 4.(vii) at $m = 2$, it follows that $u * v = \Lambda\left(g^{(0)}, g^{(1)}\right)$. It remains to show that $g^{(0)}$ and $g^{(1)}$ are *d'Alembertian. We have*

$$\begin{aligned}
(a^{(0)} * b^{(0)})_n &= \sum_{k=0}^{n} \frac{2^k k!}{2(n-k)+1} = \sum_{k=0}^{n} \frac{2^{k-1} k!}{n-k+\frac{1}{2}}, \\
(a^{(1)} * b^{(1)})_{n-1} &= \sum_{k=0}^{n} \frac{(2k+1)!}{2^k k!(2(n-k-1)+2)} = \sum_{k=0}^{n} \frac{(2k+1)!}{2^{k+1} k!(n-k)}, \\
(a^{(0)} * b^{(1)})_n &= \sum_{k=0}^{n} \frac{2^k k!}{2(n-k)+2} = \sum_{k=0}^{n} \frac{2^{k-1} k!}{n-k+1}, \\
(a^{(1)} * b^{(0)})_n &= \sum_{k=0}^{n} \frac{(2k+1)!}{2^k k!(2(n-k)+1)} = \sum_{k=0}^{n} \frac{(2k+1)!}{2^{k+1} k!(n-k+\frac{1}{2})}.
\end{aligned}$$

In an analogous way as we did it for $(a^{(0)} * b^{(0)})_n$ in Example 6, we can compute explicit d'Alembertian representations for $(a^{(1)} * b^{(1)})_{n-1}$, $(a^{(0)} * b^{(1)})_n$, and $(a^{(1)} * b^{(0)})_n$. After some additional simplification we obtain (2), (3) on p. 6.

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References

[1] Abramov, S.A., 1991. An algorithm for finding quasi-rational solutions of differential and difference equations with polynomial coefficients (Russian). *Vestnik MGU* Ser. 15 (N 1), 43–48.

[2] Abramov, S.A., 1993. On d'Alembert substitution. In: ISSAC '93. ACM, pp. 20–26.

[3] Abramov, S.A., Petkovšek, M., 1994. D'Alembertian solutions of linear operator equations. In: ISSAC '94. ACM, pp. 169–174.

[4] Abramov, S.A., Petkovšek, M., Ryabenko, A., 2000. Special formal series solutions of linear operator equations. *Discrete Math.* 210 (1–3), 3–25.

[5] Abramov, S.A., Petkovšek, M., Zakrajšek, H., 2018. Convolutions of Liouvillian sequences. arXiv:1803.08747 [cs.SC].

[6] Gosper, R.W. Jr., 1978. Decision procedure for indefinite hypergeometric summation. *Proc. Nat. Acad. Sci. U.S.A.* 75 (1), 40–42.

[7] Hendriks, P.A., Singer, M.F., 1999. Solving difference equations in finite terms. *J. Symb. Comput.* 27 (3), 239–259.

[8] Petkovšek, M., 1992. Hypergeometric solutions of linear recurrences with polynomial coefficients. *J. Symb. Comput.* 14 (2–3), 243–264.

[9] Petkovšek, M., 2006. Symbolic computations over sequences (Russian). *Programmirovanie* (2), 8–15; translation in *Program. Comput. Software* 32 (2), 65–70.

[10] Petkovšek, M., Zakrajšek, H., 2013. Solving linear recurrence equations with polynomial coefficients. In: Schneider, C., Blümlein, J., Eds.: *Computer Algebra in Quantum Field Theory: Integration, Summation and Special Functions.* Texts and Monographs in Symbolic Computation. Springer, Wien, pp. 259–284.

[11] Reutenauer, C., 2012. On a matrix representation for polynomially recursive sequences. *Electron. J. Combin.* 19 (3), P36.

[12] Stanley, R.P., 1980. Differentiably finite power series. *European J. Combin.* 1 (2), 175–188.

[13] Stanley, R.P., 1999. *Enumerative Combinatorics, Vol. 2.* Cambridge University Press, Cambridge.

[14] Zeilberger, D., 1990. A fast algorithm for proving terminating hypergeometric identities. *Discrete Math.* 80 (2), 207–211.

[15] Zeilberger, D., 1991. The method of creative telescoping. *J. Symb. Comput.* 11 (3), 195–204.