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ORDINARY
DIFFERENTIAL EQUATIONS

Truncated Series and Formal Exponential-Logarithmic Solutions of Linear Ordinary Differential Equations

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Abstract—The approach we used earlier to construct Laurent and regular solutions enables one, in combination with the well-known Newton polygon algorithm, to find formal exponential-logarithmic solutions of linear ordinary differential equations the coefficients of which have the form of truncated power series. (Thus, only incomplete information about the original equation is available.) The series involved in the solution are also represented in truncated form. For these series, the combined approach proposed enables one to obtain the maximum possible number of terms.

Keywords: linear ordinary differential equations, truncated power series, formal exponential-logarithmic solutions, Newton polygons

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1. INTRODUCTION

This article is a continuation of our works [1, 2]. Here, we again consider issues concerning the construction of solutions of linear differential equations specified approximately: the coefficients of the equations are represented by power series for which only the leading terms are known. Each coefficient is given in the form $a(x) + O(x^t)$, where $a(x)$ is a polynomial, $t > \deg a(x)$. Again we are interested in information about solutions (this time, about formal exponential-logarithmic solutions: the definition is given in Section 2.2) that does not depend on the unknown "tails" of the coefficients, i.e., information that is invariant to all possible prolongations of the available truncated series.

Thus, the original equation is specified not completely. As a result, the solution is also obtained in incomplete from, but the available initial information is used as much as possible. Namely, we are trying to ensure that the maximum possible number of terms is determined in the series involved in the solution.

The approach used in [1, 2] is based on the involvement of symbolic coefficients in the construction of solutions. Those are the symbols used to represent unspecified coefficients hidden behind the symbols O. We call these added symbolic coefficients *literals*; they can also be called *literal coefficients*. Briefly speaking, our method of literal coefficients consists in successive calculation of the coefficients of the series entering into the solutions, and these calculations use the quantities known from the equation and are performed as long as the values of the literals have no effect on the quantities that appear in the solutions. This approach is used in this article in combination with the well-known Newton polygon method [3–6].

In our article, the Newton polygon method is used to find the exponential parts, and the method of literal coefficients, to find the regular parts of the sought-for solutions; the necessary definitions are given in Section 2.2.

As already noted in [2], A.D. Bryuno in [7] proposed a method based on the Newton polygons, which enables one to find any number of terms for the series entering into the solution. This approach was further developed in [8]. The equations (in the general case, nonlinear) are defined using the explicitly specified analytic functions of one or several variables. Obviously, this is a somewhat different problem.

The aim of our approach is to obtain the maximum possible number of terms of the series entering into the solutions of the equation, which, in fact, is specified not completely: the series that specify the coefficients of the equation are known only in truncated form. This maximality is justified in this article by Propositions 1, 2, and 3.

Section 6 describes our implementation of the algorithm proposed in Section 5. Examples of its use are presented. The algorithm was implemented using the Maple 2019 software [9].

2. PRELIMINARY INFORMATION

2.1. Basic Concepts

Let K be an algebraically closed field of characteristic 0. We will use the following standard notation:

- $K[x]$ is the ring of polynomials in x over K,
- $K[[x]]$ is the ring of formal power series in x over K,
- $K((x))$ is the field of formal Laurent series in x over K, which is the field of quotients of the ring $K[[x]]$.

The degree $\deg a(x)$ of a polynomial $a(x)$ is determined as usual, and $\deg 0 = -\infty$. For elements of the ring $K[[x]]$ and the field $K((x))$, the concept of valuation is introduced: for $a(x) = \sum a_i x^i$, we set

$$\operatorname{val} a(x) = \min\{i \,|\, a_i \neq 0\},$$

and $\operatorname{val} 0 = \infty$.

In $K[x]$, $K[[x]]$, and $K((x))$, the differentiation $D = \frac{d}{dx}$ is defined. We will consider operators and differential equations written using the operation $\theta = x\frac{d}{dx}$. In the original operator

$$L = \sum_{i=0}^{r} a_i(x)\theta^i \in K[x][\theta] \tag{1}$$

the polynomial coefficient $a_i(x)$ will be assumed below to have the form

$$a_i(x) = \sum_{j=0}^{t_i} a_{ij}x^j,$$

where t_i is a nonnegative integer greater than or equal to $\deg a_i(x)$, $i = 0, 1, \ldots, r$ (if $t_i > d_i = \deg a_i(x)$, then $a_{ij} = 0$ for $j = d_i + 1, d_i + 2, \ldots, t_i$). The powers of the operator θ in (1) are understood as compositions $\theta^i y = \theta(\theta(\ldots\theta(y)\ldots))$ with i left and right parentheses.

Henceforward, we put into correspondence to the *truncated differential equation*

$$\left(a_r(x) + O(x^{t_r+1})\right)\theta^r y(x) + \ldots + \left(a_1(x) + O(x^{t_1+1})\right)\theta y(x) + \left(a_0(x) + O(x^{t_0+1})\right)y(x) = 0, \tag{2}$$

$t_i \geq \deg a_i(x)$, $i = 0, 1, \ldots, r$, operator (1) and a set of numbers $t_0, t_1, \ldots, t_r$. An operator

$$\sum_{i=0}^{r} a_i(x)\theta^i \in K[[x]][\theta],$$

with the coefficients represented by series

$$a_i(x) = \sum_{j=0}^{\infty} a_{i,j}x^j \in K[[x]],$$

will be denoted by letter $\mathscr{L}$. An operator with polynomial coefficients will be denoted by letter L.

Definition 1. A *prolongation* of Eq. (2) will be understood as any equation $\mathscr{L}(y) = 0$ with the operator

$$\mathscr{L} = \sum_{i=0}^{r} \tilde{a}_i(x)\theta^i \in K[[x]][\theta],$$

for which $\tilde{a}_i(x) - a_i(x) = O(x^{t_i+1})$, i.e., $\operatorname{val}(\tilde{a}_i(x) - a_i(x)) > t_i$, $i = 0, 1, \ldots, r$.

If L (or $\mathscr{L}$) is some differential operator, then the *solutions of the operator* L (or $\mathscr{L}$) will be understood as the solutions of the equation $L(y) = 0$ (correspondingly, $\mathscr{L}(y) = 0$).

2.2. Formal Exponential-Logarithmic Solutions

The notation $\mathbb{Z}_{>0}$ is used below for the set of positive integers.

Definition 2. Formal *exponential-logarithmic* solutions of a differential equation are solutions of the form

$$\exp\{Q(x)\}x^{\lambda}w(x), \tag{3}$$

where $Q(x) \in K[x^{-1/q}]$, $q \in \mathbb{Z}_{>0}$, $\lambda \in K$, and $w(x) \in K((x^{1/q}))[\ln(x^{1/q})]$, as well as finite sums of solutions of form (3).

We will call $\exp\{Q(x)\}$ the *exponential part* of formal solution (3); correspondingly, $Q(x)$ will be called the *exponent of the exponential part* and $x^{\lambda}w(x)$, the *regular part* of formal solution (3). In [3, 4], an algorithm for constructing r linearly independent formal exponential-logarithmic solutions of form (3) for $\mathscr{L}(y) = 0$ was discussed.

In the Introduction, we have already outlined an approach based on the use of literal coefficients or, as we agreed to call them in short, literals. It was said that, in this article we are using this approach in combination with Newton polygon method. It is therefore appropriate to provide some general information about this method. In this section, we assume that the equation is completely defined, i.e., in our context, that the series representing the coefficients of the equation are fully known to us. Let us give the definition of the Newton polygon from [3, 4]: our consideration below is based on this definition of the Newton polygon. (A slightly different definition is used in [7, 8].)

Definition 3. Suppose that, in the plane $\mathbb{R}^2$, for $\mathscr{L}(y) = 0$, the points (i, n_i) are marked, where $n_i = \operatorname{val} a_i(x)$, $i = 0, 1, \dots, r$, and the set $\mathfrak{Q}_i^+ = \{(x, y) \in \mathbb{R}^2,\ x \le i,\ y \ge n_i\}$ is defined. Let M^+ be the union of all $\mathfrak{Q}_i^+$ for $i = 0, 1, \dots, r$. The *Newton polygon* for $\mathscr{L}(y) = 0$ is the convex hull of the set M^+. This polygon will be denoted by $\mathcal{N}(\mathscr{L})$.

Let $\mathcal{N}(\mathscr{L})$ have s vertices (i_j, η_j), $j = 1, \dots, s$, where $i_{j-1} < i_j$ for $j > 1$, $i_0 = 0$. We denote the side with the vertices (i_{j-1}, η_{j-1}) and (i_j, η_j) by S_j and assign to it its *slope* $k_j = (\eta_j - \eta_{j-1})/(i_j - i_{j-1})$. Let us give a brief description of the algorithm from [3, 4] for constructing formal exponential-logarithmic solutions for $\mathscr{L}(y) = 0$.

If $\mathcal{N}(\mathscr{L})$ has a side with a slope 0, then the side S_1 has vertices $(0, \eta_1)$ and (i_1, η_1), where $i_1 > 0$. Then, the original equation has i_1 solutions of form (3), in which $q = 1$ and $Q = 0$. Such solutions are called *regular*. The classical algorithms for constructing regular solutions can be found, e.g., in [10; 11, Ch. II, VIII; 12, Ch. IV]. Such algorithms were also proposed in [7].

If some side S_j has a slope $k > 0$, then (see, e.g., [4, Ch. 3]) the original equation has $i_j - i_{j-1}$ different formal solutions with the exponent of the exponential part

$$Q(x) = -\frac{\varepsilon}{kx^k} + \cdots,$$

where $\varepsilon \in K\backslash\{0\}$; the ellipsis replaces here a finite number of terms containing smaller than k rational degrees of x^{-1}. The number ε is a root of the *characteristic equation* associated with the side S_j:

$$\sum_{\substack{i=i_{j-1} \\ (i,n_i)\in S_j}}^{i_j} a_{i,n_i}\varepsilon^{i-i_{j-1}} = 0, \tag{4}$$

where the notation $(i, n_i) \in S_j$ means that the point (i, n_i) belongs to the side S_j. We will call $-\varepsilon/(kx^k)$ the *leading term* of $Q(x)$.

Suppose that the leading terms in the exponential parts of all formal solutions are specified via $\mathcal{N}(\mathscr{L})$, i.e., $-\varepsilon_{j,l}/(k_j x^{k_j})$, $l = 1, \dots, i_j - i_{j-1}$, $j = 1, \dots, s$, are defined. For each nonzero slope $k_j = p_j/q_j$

($\gcd(p_j, q_j) = 1$, i.e., p_j and q_j are coprime) and each root $\varepsilon_{j,l}$ of characteristic equation (4) in the original equation, we make the substitution

$$y(x) = \exp\left\{-\frac{\varepsilon_{j,l}}{k_j t^{p_j}}\right\} z(t), \quad x = t^{q_j}, \tag{5}$$

where $z(t)$ is a new unknown and t is a new independent variable. After reducing by $\exp\{-\varepsilon_{j,l}/(k_j t^{p_j})\}$, we obtain the equation $\mathcal{L}_1(z) = 0$ of order r with coefficients from $K((t))$. Denote by $\operatorname{val}\mathcal{L}_1$ the minimum value of valuations of all coefficients of $\mathcal{L}_1$. If $\operatorname{val}\mathcal{L}_1 < 0$, we multiply the coefficients of the equation by $t^{-\operatorname{val}\mathcal{L}_1}$ to obtain the equation with the coefficients from $K[[t]]$. We apply to the new equation the algorithm for constructing formal exponential-logarithmic solutions, but, in this case, we will consider in $\mathcal{N}(\mathcal{L}_1)$ only those sides whose slope is smaller than p_j. For each thus constructed solution $z(t)$ of the equation $\mathcal{L}_1(z) = 0$, we obtain the following solution of the original equation $\mathcal{L}(y) = 0$:

$$y(x) = \exp\left\{-\frac{\varepsilon_{j,l}}{k_j x^{k_j}}\right\} z(x^{1/q_j}).$$

Remark 1. In [5, 6], an efficient algorithm is proposed for constructing formal solutions of the equation $\mathcal{L}(y) = 0$, where the field K in the general case is not algebraically closed. This algorithm, call by the authors rational Newton's algorithm, constructs solutions in the form

$$y(x) = \exp\{Q(1/t)\} t^{\lambda} w(t), \quad x = \Lambda t^{q}, \tag{6}$$

where $Q(1/t) \in K_1[1/t]$, $q \in \mathbb{Z}_{>0}$, $\lambda, \Lambda \in K_1$, and $w(t) \in K_1[[t]][\ln(t)]$. The field K_1 is a finite algebraic extension of K. For $q > 1$, formula (6) gives at least q different solutions of form (3). For each side $\mathcal{N}(\mathcal{L})$ with a slope $k_j = p_j/q_j$, with which the characteristic equation $\chi(\varepsilon) = 0$ is associated, substitutions (5) are replaced with the substitutions

$$y(x) = \exp\left\{-\frac{\varepsilon}{k_j t^{p_j}}\right\} z(t), \quad x = \Lambda t^{q_j},$$

where ε is the root of the divisor irreducible over K of the polynomial $\chi(\varepsilon)$. In the computer implementation of our algorithm, we use the algorithm from [6].

3. EXPONENTIAL PART OF THE SOLUTION OF AN EQUATION WITH TRUNCATED COEFFICIENTS

For Eq. (2) with truncated coefficients, we set $n_i = \operatorname{val} a_i(x)$, $i = 0, 1, \ldots, r$. Suppose that the valuation of all coefficients of this equation are the same for all prolongations, i.e., $n_i < \infty$ and $a_i(x) \neq 0$ for $i = 0, 1, \ldots, r$. Then, for any prolongation $\mathcal{L}(y) = 0$ of Eq. (2) $\mathcal{N}(\mathcal{L})$ and the characteristic equations associated with its sides will also coincide with $\mathcal{N}(L)$ and the characteristic equations for $\mathcal{N}(L)$, where L is defined by (1). For all prolongations of Eq. (2), the sets of all leading terms of the exponential parts of the formal solutions will be the same.

Example 1. Consider the truncated equation

$$\begin{aligned}(-x^2 + O(x^4))\theta^5 y(x) + (x^3 + O(x^4))\theta^4 y(x) + (x + O(x^3))\theta^3 y(x)\\ + (x^3 + O(x^4))\theta^2 y(x) + (x + O(x^3))\theta y(x) + (-1 + O(x))y(x) = 0.\end{aligned}$$

Here,

$$L = -x^2\theta^5 + x^3\theta^4 + x\theta^3 + x^3\theta^2 + x\theta - 1.$$

The polygon $\mathcal{N}(\mathcal{L})$ for each prolongation $\mathcal{L}(y) = 0$ of this truncated equation has vertices $(0,0)$, $(3,1)$, and $(5,2)$. The set of all leading terms of the exponential parts of the formal solutions for each prolongation is as follows:

$$-\frac{3}{x^{1/3}}, \quad \frac{3(-1)^{1/3}}{x^{1/3}}, \quad -\frac{3(-1)^{2/3}}{x^{1/3}}, \quad -\frac{2}{x^{1/2}}, \quad \frac{2}{x^{1/2}}.$$

Thus, the formal solutions for each prolongation have the form

$$\exp\left\{-\frac{3}{x^{1/3}}\right\}y_1(x), \quad \exp\left\{\frac{3(-1)^{1/3}}{x^{1/3}}\right\}y_2(x), \quad \exp\left\{-\frac{3(-1)^{2/3}}{x^{1/3}}\right\}y_3(x),$$
$$\exp\left\{-\frac{2}{x^{1/2}}\right\}y_4(x), \quad \exp\left\{\frac{2}{x^{1/2}}\right\}y_5(x),$$

where the factors $y_1(x)$, $y_2(x)$, $y_3(x)$, $y_4(x)$, $y_5(x)$ are so far unknown.

Now suppose that, in Eq. (2), $a_i(x) = 0$ and $n_i = \infty$ for some i. Let all t_i be specified so that the points (i, t_i) lie inside $\mathcal{N}(L)$. In this case, as in the previous one, $\mathcal{N}(\mathcal{L})$ for each prolongation $\mathcal{L}(y) = 0$ of Eq. (2) and the characteristic equations associated with the sides $\mathcal{N}(\mathcal{L})$ will coincide with $\mathcal{N}(L)$ and the characteristic equations for L. For all prolongations, the sets of all leading terms of the exponential parts of the formal solutions will be the same.

Example 2. For a truncated equation

$$(-x^2 + O(x^4))\theta^5 y(x) + (x^3 + O(x^4))\theta^4 y(x) + (x + O(x^3))\theta^3 y(x)$$
$$+ O(x^3)\theta^2 y(x) + O(x)\theta y(x) + (-1 + O(x))y(x) = 0$$

we have

$$L = -x^2\theta^5 + x^3\theta^4 + x\theta^3 - 1.$$

The Newton polygon coincides with the Newton polygon from Example 1. The set of all leading terms of the exponential parts of the formal solutions for all prolongations is the same as in Example 1.

If, in Eq. (2) the valuation of the leading coefficient may be different for different prolongations (i.e., $n_r = \infty$ and $a_r(x) = 0$ in (2)), then there exists a prolongation $\mathcal{L}_1(y) = 0$ of Eq. (2) such that its order is lower than r and a prolongation $\mathcal{L}_2(y) = 0$ the order of which is equal to r and the valuation of the leading coefficient can be either $t_r + 1$ or any integer greater than t_r. Thus, the last side in $\mathcal{N}(\mathcal{L}_2)$ and the corresponding leading coefficients of the exponential part are not invariant to $\mathcal{L}_1(y) = 0$.

Example 3. Consider a truncated equation

$$O(x^2)\theta^5 y(x) + (x^3 + O(x^4))\theta^4 y(x) + (x + O(x^3))\theta^3 y(x)$$
$$+ O(x^3)\theta^2 y(x) + O(x)\theta y(x) + (-1 + O(x))y(x) = 0.$$

For

$$L = x^3\theta^4 + x\theta^3 - 1$$

the set of all leading terms of the exponential parts of formal solutions will be

$$-\frac{3}{x^{1/3}}, \quad \frac{3(-1)^{1/3}}{x^{1/3}}, \quad -\frac{3(-1)^{2/3}}{x^{1/3}}, \quad \frac{1}{2x^2},$$

and, for the prolongation

$$\tilde{L} = -x^2\theta^5 + x^3\theta^4 + x\theta^3 - 1,$$

the set of all leading terms will be the same as in Example 1. Note that, in this example, the first side of the Newton polygon will be the same for all prolongations, and, therefore, all prolongations will have solutions of the form

$$\exp\left\{-\frac{3}{x^{1/3}}\right\}y_1(x), \quad \exp\left\{\frac{3(-1)^{1/3}}{x^{1/3}}\right\}y_2(x), \quad \exp\left\{-\frac{3(-1)^{2/3}}{x^{1/3}}\right\}y_3(x).$$

Thus, in some cases, it is possible, knowing only truncated equation (2), to construct the leading terms of the exponents of the exponential part of some formal solutions that are invariant to all prolongations of Eq. (2). Suppose that, in Eq. (2), for some t_i, the point $(i, t_i + 1)$ lies outside $\mathcal{N}(L)$, where L is defined by (1). Then, there are prolongations of Eq. (2) such that their Newton polygons will differ from $\mathcal{N}(L)$, but $\mathcal{N}(L)$ will be a subset for each of them. At the same time, they will all be a subset of $\mathcal{N}(\tilde{L})$, where

$$\tilde{L} = L + \sum_{\substack{i=0 \\ (i,t_i)\notin\mathcal{N}(L)}}^{r} x^{t_i+1}\theta^i. \tag{7}$$

Let us construct $\mathcal{N}(L)$ and $\mathcal{N}(\tilde{L})$ and determine which of their sides coincide. These and only these sides will be invariant to all prolongations of Eq. (2), and, if the associated characteristic equations are the same, then the corresponding coefficients of the leading terms will be invariant to all prolongations of Eq. (2).

Example 4. Consider a truncated equation

$$\begin{gathered}(-x^3 + O(x^4))\theta^5 y(x) + (x^3 + O(x^4))\theta^4 y(x) + (x + O(x^3))\theta^3 y(x) \\ + O(x^3)\theta^2 y(x) + O(1)\theta y(x) + (-1 + O(x))y(x) = 0.\end{gathered} \tag{8}$$

Then, we have

$$L = -x^3\theta^5 + x^3\theta^4 + x\theta^3 - 1,$$

$$\tilde{L} = -x^3\theta^5 + x^3\theta^4 + x\theta^3 + x^3\theta^2 + \theta - 1.$$

In $\mathcal{N}(L)$ and $\mathcal{N}(\tilde{L})$, the common side has vertices $(3,1)$ and $(5,3)$. It corresponds to the leading terms

$$-\frac{1}{x}, \quad \frac{1}{x}.$$

These leading terms and only they are invariant to all prolongations of Eq. (8).

Proposition 1. *Suppose that, for all prolongations of Eq.* (2), *the corresponding Newton polygons have a common side S with the vertices $(i', n_{i'})$ and $(i'', n_{i''})$, $n_{i''} \neq n_{i'}$, and this side has a slope k. Let there be i, $i' \le i \le i''$, such that $(i, t_i + 1) \in S$. Then, there is an prolongation of Eq.* (2) *for which the leading terms of the exponents of the exponential parts of all its solutions differ from $-\varepsilon/(kx^k)$, where ε is the root of the characteristic equation for* (1), *associated with the side S.*

Proof. Consider two prolongations of Eq. (2): $L(y) = 0$ and $L_1(y) = 0$, where L has form (1) and

$$L_1 = L + x^{t_i+1}\theta^i.$$

The characteristic equations $P(\varepsilon) = 0$ for $L(y) = 0$ and, $P_1(\varepsilon) = 0$ for $L_1(y) = 0$, associated with the side S, are different:

$$P(\varepsilon) = \sum_{\substack{j=i' \\ (j,n_j)\in S}}^{i''} a_{j,n_j}\varepsilon^{j-i'}, \qquad P_1(\varepsilon) = P(\varepsilon) + \varepsilon^{i-i'}.$$

We see that $P(0) \neq 0$, $P_1(0) \neq 0$, and $P(\varepsilon) - P_1(\varepsilon) = \varepsilon^{i-i'}$. Thus, $P(\varepsilon)$ and $P_1(\varepsilon)$ have no common roots, which proves the proposition.

4. REGULAR PART OF THE SOLUTION

The regular part of the solution is calculated using the algorithm proposed in [2] for constructing truncated regular solutions of differential equations with coefficients in the form of truncated series. The regular solution can be written as

$$x^\lambda \sum_{s=0}^{k} g_{k-s}(x)\frac{\ln^s x}{s!},$$

where $k \in \mathbb{Z}_{\ge 0}$ and $g_j(x) \in K((x))$, $j = 0, 1, \ldots, k$. For Eq. (2), the algorithm from [2] constructs regular solutions with maximum truncations of the series $g_{k-s}(x)$ entering into them, such that the solutions are invariant to various possible prolongations of the equation (the maximality of truncation means that adding terms of higher degree to any of these truncated series entails a loss of invariance to possible prolongations of the equation). The set of possible values of λ is determined based on the roots of the characteristic equation associated with the side of Newton polygon with slope 0, after which the search for truncated regular solutions reduces to finding truncated solutions in $K((x))$ (Laurent solutions), which, in turn, is performed using the algorithm from [1]. The truncated regular solutions found contain arbitrary constants.

The algorithms from [1, 2] assume that the constant term of at least one of $a_0(x),\ldots,a_r(x)$ is nonzero. This guarantees that the characteristic equation is invariant to prolongations of the original equation. If this assumption is not satisfied, then there are no invariant truncations of regular solutions. This follows from the next proposition.

Proposition 2. *Let an equation of form* (2) *be associated with an operator L of form* (1) *and a set of numbers $t_0,\ldots,t_r$ as described in Section* 2.1. *Let $\mathcal{N}(L)$ have a vertex $(0,\eta_1)$. Let $t_i<\eta_1$ for some t_i, $0\le i\le r$. Then, for any $\lambda\in K$, there is a prolongation of Eq.* (2) *that does not have a regular solution of the form*

$$x^{\lambda}(\ln^k x)(1+O(x))+x^{\lambda}u(x), \tag{9}$$

where $k\in\mathbb{Z}_{\ge 0}$, $u(x)\in K((x))[\ln(x)]$, and the degree of $u(x)$ as a polynomial in $\ln x$ *is lower than k.*

Proof. If the equation has a solution of form (9), then the characteristic equation associated with the side with slope 0 of the Newton polygon has a root λ. Let us consider three cases.

1. $t_0+1<\eta_1$. Then, $\mathcal{N}(\mathcal{L})$ has a side with vertices $(0,\eta_1)$, (i_1,η_1), $i_1>0$; therefore, $L(y)=0$ has regular solutions. For $L_2(y)=0$, where

$$L_2=L+x^{t_0+1},$$

the first side $\mathcal{N}(L_2)$ has vertices $(0,t_0+1)$ and (i_1,η_1); i.e., its slope is not 0 and, therefore, $L_2(y)=0$ has no regular solutions.

2. $t_0+1=\eta_1$. Then, $\mathcal{N}(L)$ and $\mathcal{N}(L_2)$ have the same side with a slope 0 with the vertices $(0,\eta_1)$ and (i_1,η_1). But the characteristic equations $P(\varepsilon)=0$ (for $L(y)=0$) and $P_2(\varepsilon)=0$ (for $L_2(y)=0$) associated with this side are different:

$$P(\varepsilon)=\sum_{j=0}^{i_1}a_{j,\eta_1}\varepsilon^j,\qquad P_2(\varepsilon)=1+P(\varepsilon).$$

Since $P_2(\varepsilon)-P(\varepsilon)=1$, $P(\varepsilon)$ and $P_2(\varepsilon)$ do not have equal roots. This means that $\lambda_1\ne\lambda_2$ for all solutions $x^{\lambda_1}\ln^{k_1}(x)(1+O(x))+\ldots$ of the first equation and solutions $x^{\lambda_2}\ln^{k_2}(x)(1+O(x))+\ldots$ of the second equation.

3. $t_0+1>\eta_1$ (hence, $t_i\ne t_0$). For $L(y)=0$ and $L_3(y)=0$, where

$$L_3=L+x^{\eta_1}\theta^i,$$

$\mathcal{N}(L)$ and $\mathcal{N}(L_3)$ have a side with a slope 0 with the same vertex $(0,\eta_1)$. For $L_3(y)=0$, this side is associated with the characteristic equation

$$P_3(\varepsilon)=\varepsilon^i+P(\varepsilon).$$

Since $P_3(\varepsilon)-P(\varepsilon)=\varepsilon^i$, $P(\varepsilon)$ and $P_3(\varepsilon)$ can have only one common root: $\varepsilon=0$. This means that, if $\lambda_1\ne 0$ and $\lambda_2\ne 0$, then $\lambda_1\ne\lambda_2$ for all solutions $x^{\lambda_1}\ln^{k_1}(x)(1+O(x))+\ldots$ of the first equation and solutions $x^{\lambda_2}\ln^{k_2}(x)(1+O(x))+\ldots$ of the second equation.

If $n_0\ne 0$, then $P(\varepsilon)$ does not have the root 0. Therefore, $\lambda_1\ne\lambda_2$ for all solutions $x^{\lambda_1}\ln^{k_1}(x)(1+O(x))+\ldots$ of the first equation and solutions $x^{\lambda_2}\ln^{k_2}(x)(1+O(x))+\ldots$ of the second equation.

If $n_0=0$, then $P(0)=0$. In this case, $L(y)=0$ has regular solutions (9) with $\lambda=0$, if and only if 0 is the maximum integer root of $P(\varepsilon)$. Consider the prolongation of Eq. (2) $L_4(y)=0$, where

$$L_4=L+\alpha x^{\eta_1}\theta^i$$

and $\alpha \in K$ is such that its characteristic equation associated with a side with slope 0,

$$P_4(\varepsilon) = \alpha\varepsilon^i + P(\varepsilon),$$

has the root 1, i.e., $P_4(1) = 0$. It is clear that

$$\alpha = -\sum_{j=0}^{i_1} a_{j,\eta_1}.$$

The equation $L_4(y) = 0$ has regular solutions (9) with $\lambda = 1$ and does not have such solutions with $\lambda = 0$.

Example 5. Consider a truncated equation

$$O(x^2)\theta^2 y(x) + (x^2 + O(x^3))\theta y(x) + O(x^3)y(x) = 0.$$

Here, $\eta_1 = 2$ and $t_2 = 1$. For this equation, there are no λ and k such that all prolongations have solutions of form (9). The prolongation

$$x^2\theta y(x) + x^3 y(x) = 0$$

has the solution $1 + O(x)$ $(\lambda = 0,\ k = 0)$. The prolongation

$$-x^2\theta^2 y(x) + x^2\theta y(x) + x^3 y(x) = 0$$

does not have solutions of this form. Its solutions are

$$x\ln(x)(1 + O(x)) + (1 + O(x)) \qquad (\lambda = 1,\ k = 1)$$

and

$$x(1 + O(x)) \qquad (\lambda = 1,\ k = 0).$$

5. THE ALGORITHM

Let us describe a recursive algorithm $\mathcal{A}$ for constructing the exponents of the exponential parts and beginnings of regular parts of formal solutions that are invariant to all prolongations of the given truncated equation.

Step 1. Input data: differential operator (1) with polynomial coefficients, integers $t_0, t_1, \dots, t_r$, and number P (at the first iteration, $P = \infty$).

Step 2. Construct $\mathcal{N}(L)$ and $\mathcal{N}(\tilde{L})$ for (1) and (7) and the set $\mathcal{S}$ of all their identical sides with a slope smaller than P, such that the conditions of Propositions 1 and 2 are not satisfied; i.e., for each side $S \in \mathcal{S}$, we have $(i, t_i + 1) \notin S$, $i = 0, \dots, r$.

Step 3. If $\mathcal{S} = \varnothing$, then

• if the first vertices $\mathcal{N}(L)$ and $\mathcal{N}(\tilde{L})$ coincide, then all prolongations do not have regular solutions and the result is NULL;

• otherwise, no invariant information about exponential-logarithmic solutions of prolongations exists; the result is FAIL.

Step 4. Set $\mathcal{Y} = \varnothing$.

Step 5. If $\mathcal{S}$ contains a side with slope 0, then, for truncated equation (2), construct by the algorithm from [2] regular solutions with maximum invariant truncations of the series in them and add the result to $\mathcal{Y}$.

Step 6. For each side $S \in \mathcal{S}$ with a nonzero slope $k = p/q$ $(\gcd(p, q) = 1)$, construct the characteristic equation associated with S and find the set of all its roots, $\mathcal{E}$. For each $\varepsilon \in \mathcal{E}$, make the substitution

$$y(x) = \exp\left\{-\frac{\varepsilon}{kt^p}\right\} z(t), \qquad x = t^q$$

in Eq. (2) and obtain a new equation with truncated coefficients. Apply to the new equation the algorithm $\mathcal{A}$ with $P = p$. If the result is FAIL or NULL, add $\exp\{-\varepsilon/(kx^k)\}Y$ to $\mathcal{Y}$; otherwise, for each element $r(t)$ from the resulting set, add $\exp\{-\varepsilon/(kx^k)\}r(x^{1/q})$ to $\mathcal{Y}$.

Step 7. The result is the set $\mathcal{Y}$.

Example 6. Applying the algorithm $\mathcal{A}$ to the truncated equation from Example 1, we obtain the following set of five elements:

$$\exp\left\{-\frac{3}{x^{1/3}}\right\}\left(c_1x^{2/3}-\frac{16c_1}{9}x+O(x^{4/3})\right),$$

$$\exp\left\{\frac{3(-1)^{1/3}}{x^{1/3}}\right\}\left(c_1x^{2/3}-\frac{16c_1(-1)^{2/3}}{9}x+O(x^{4/3})\right),$$

$$\exp\left\{-\frac{3(-1)^{2/3}}{x^{1/3}}\right\}\left(c_1x^{2/3}+\frac{16c_1(-1)^{1/3}}{9}x+O(x^{4/3})\right),$$

$$\exp\left\{-\frac{2}{x^{1/2}}\right\}\left(c_1x^{5/4}+\frac{15c_1}{16}x^{7/4}+O(x^{9/4})\right),$$

$$\exp\left\{\frac{2}{x^{1/2}}\right\}\left(c_1x^{5/4}-\frac{15c_1}{16}x^{7/4}+O(x^{9/4})\right),$$

where c_1 is an arbitrary constant generated by the algorithm for constructing truncated regular solutions from [2].

To complete this example, we rename the arbitrary constants and compose a general truncated formal exponential-logarithmic solution:

$$\exp\left\{-\frac{3}{x^{1/3}}\right\}\left(c_1x^{2/3}-\frac{16c_1}{9}x+O(x^{4/3})\right)+\exp\left\{\frac{3(-1)^{1/3}}{x^{1/3}}\right\}\left(c_2x^{2/3}-\frac{16c_2(-1)^{2/3}}{9}x+O(x^{4/3})\right)$$
$$+\exp\left\{-\frac{3(-1)^{2/3}}{x^{1/3}}\right\}\left(c_3x^{2/3}+\frac{16c_3(-1)^{1/3}}{9}x+O(x^{4/3})\right)+\exp\left\{-\frac{2}{x^{1/2}}\right\}\left(c_4x^{5/4}+\frac{15c_4}{16}x^{7/4}+O(x^{9/4})\right)$$
$$+\exp\left\{\frac{2}{x^{1/2}}\right\}\left(c_5x^{5/4}-\frac{15c_5}{16}x^{7/4}+O(x^{9/4})\right).$$

At each iteration of the recursive algorithm $\mathcal{A}$, when constructing the leading term of the exponential part of the formal solution, we use the sides $\mathcal{N}(L)$ for (1), which are invariant to all prolongations along with the characteristic equations associated with them. When constructing the regular part, we use the algorithm from [2], which obtains the maximum possible number of terms of the series. All this gives us to the following proposition.

Proposition 3. *Suppose that the operator L has form* (1), $t_i \geq \deg a_i(x)$, $i = 0,1,\dots,r$, *and the application of the algorithm $\mathcal{A}$ to truncated equation* (2) *made it possible to conclude that, for any prolongation of this equation, there is a formal exponential-logarithmic solution with the exponent $Q(x) \in K[x^{-1/q}]$ of the exponential part. Suppose that the substitution $y(x) = \exp\{Q(x)\}z(t)$, $x = t^q$, gave a new equation with truncated coefficients, for which regular solutions were found by the algorithm from* [2]. *Then, each of the resulting formal exponential-logarithmic solutions of Eq.* (2) *contains truncated series with the maximum possible number of terms invariant to the prolongations of Eq.* (2).

6. IMPLEMENTATION AND EXAMPLES OF USE

The algorithm proposed above for constructing the invariant part of formal solutions for a truncated differential equation was implemented in the Maple 2019 environment in the form of procedure `FormalSolution` as an extension of the `TruncatedSeries` package presented in [2, 13]. The package and the `Maple` session with examples of using its procedures are available at http://www.ccas.ru/ca/TruncatedSeries.

The first argument of the procedure is a differential equation of form (2). Application of θ^i to an unknown function $y(x)$ is written as `theta(y(x),x,i)`. The truncated coefficients of the equation are given in the form of expressions $a_i(x) + O(x^{t_i+1})$, where $a_i(x)$ is a polynomial of degree not higher than t_i over the field $\overline{\mathbb{Q}}$, i.e., over the field of algebraic numbers.

The second argument of the procedure is the unknown function, e.g., `y(x)`.

The result of the procedure is as follows:

• Maple constant `FAIL` if there are no invariant initial parts of solutions of the given equation;

• Maple constant `NULL` if there are no invariant initial parts of solutions of the given equation and no prolongation of this equation has regular solutions;

• a list of truncated formal solutions that are invariant to prolongation of the given equation.

The truncated formal solution is a finite sum of expressions of the form $\_c_j e^Q y_i(x)$ and/or $e^Q R$, where

• $Q \in \bar{\mathbb{Q}}[x^{-1/q}]$, where $q \in \mathbb{Z}_{>0}$ is the invariant part of the exponent of the exponential part of the formal solution;

• $\_c_j$, where $j \in \mathbb{Z}_{>0}$ denotes an arbitrary constant;

• $y_i(x)$, where $i \in \mathbb{Z}_{>0}$ denotes the part of the formal solution that is not invariant to all prolongations of the given equation;

• R is a finite sum of expressions of the form

$$x^{\lambda}\sum_{s=0}^{k}\left(g_{k-s}+O(x^{m_s/q})\right)\frac{\ln^s x^{1/q}}{s!},$$

where $\lambda \in \bar{\mathbb{Q}}$, $k \in \mathbb{Z}_{\geq 0}$, $g_{k-s} \in \bar{\mathbb{Q}}[\_c_1, \_c_2, \ldots][x^{1/q}]$, $m_s \in \mathbb{Z}_{>0}$, for $s = 0, 1, \ldots, k$.

Algebraic numbers are represented using the standard Maple construct `RootOf`. In the following example, $RootOf(\_Z^2+2, index=1) = \sqrt{-2}$ and $RootOf(\_Z^2+2, index=2) = -\sqrt{-2}$. (The presence of similar constructs in the original differential equations is also possible.)

```
> (x+O(x^3))*theta(y(x),x,2)+(x^2+O(x^3))*theta(y(x),x,1)+
(2+O(x^2))*y(x):
> FormalSolution(%,y(x));
```

$$\left[e^{-\frac{2RootOf(\_Z^2+2,index=1)}{\sqrt{x}}}x^{1/4}\left(\_c_1+\frac{\sqrt{x}\_c_1 RootOf(\_Z^2+2,index=1)}{32}-\frac{521x\_c_1}{1024}+O(x^{3/2})\right)\right.$$
$$\left.+\,e^{-\frac{2RootOf(\_Z^2+2,index=2)}{\sqrt{x}}}x^{1/4}\left(\_c_2+\frac{\sqrt{x}\_c_1 RootOf(\_Z^2+2,index=2)}{32}-\frac{521x\_c_1}{1024}+O(x^{3/2})\right)\right]$$

The following equation has solutions containing both an exponential and logarithms:

```
> (x^2+x^5+O(x^6))*theta(y(x),x,2)+(2*x+x^4+O(x^5))*theta(y(x),x,1)+
(1-x+x^3+O(x^4))*y(x):
> FormalSolution(%, y(x));
```

$$e^{\frac{1}{x}}\left(\_c_2+x(-\_c_2+2\_c_1)+O(x^2)+\ln(x)\left(-x\_c_1+\_c_1+O(x^2)\right)\right)$$

Let us illustrate the work of procedure `FormalSolution` by a few more examples.

1. An equation the solutions of which has no invariant initial parts:

```
> O(x^4)*theta(y(x),x,2)+O(x)*theta(y(x),x,1)+O(1)*y(x):
> y(x)=FormalSolution(%,y(x));
```

$$FAIL$$

2. Let us add some terms to the last coefficient of the previous equation. The procedure returns the Maple constant `NULL`, i.e., all continuations have no regular solutions. In the Maple 2019 session, the result has the form

```
> O(x^4)*theta(y(x),x,2)+O(x)*theta(y(x),x,1)+(2+O(x^2))*y(x):
> y(x)=FormalSolution(%,y(x));
```

$$y(x) = ()$$

Below, in examples 3–8, we continue to add new terms to the coefficients (“refine the coefficients”) of the original equation.

3. As a coefficient at θy, we take $3x + O(x^2)$:

```
> O(x^4)*theta(y(x),x,2)+(3*x+O(x^2))*theta(y(x),x,1)+
(2+O(x^2))*y(x):
> FormalSolution(%,y(x));
```

$$\left[\_c_1 e^{\frac{2}{3x}} y_1(x)\right]$$

4. Now, as a coefficient at θy, we take $3x + O(x^3)$:

```
> O(x^4)*theta(y(x),x,2)+(3*x+O(x^3))*theta(y(x),x,1)+
(2+O(x^2))*y(x):
> FormalSolution(%,y(x));
```

$$\left[e^{\frac{2}{3x}}(\_c_1 + O(x))\right]$$

5. In the previous version, we additionally refine the leading coefficient: we take it equal to $4x^4 + O(x^5)$:

```
> (4*x^4+O(x^5))*theta(y(x),x,2)+(3*x+O(x^3))*theta(y(x),x,1)+
(2+O(x^2))*y(x):
> FormalSolution(%,y(x));
```

$$\left[e^{\frac{2}{3x}}(\_c_1 + O(x)) + \_c_2 e^{\frac{1}{4x^3}} y_1(x)\right]$$

6. Once again we refine the leading coefficient: we take it equal to $4x^4 + O(x^8)$:

```
>  (4*x^4+O(x^8))*theta(y(x),x,2)+(3*x+O(x^4))*theta(y(x),x,1)+
(2+O(x^2))*y(x):
> FormalSolution(%,y(x));
```

$$\left[e^{\frac{2}{3x}}(\_c_1 + O(x)) + \_c_2 e^{\frac{1}{4x^3} - \frac{2}{3x}} y_1(x)\right]$$

7. We additionally refine the coefficient at θy: we take it equal to $3x + O(x^5)$:

```
> (4*x^4+O(x^8))*theta(y(x),x,2)+(3*x+O(x^5))*theta(y(x),x,1)+
(2+O(x^2))*y(x):
> FormalSolution(%,y(x));
```

$$\left[e^{\frac{2}{3x}}(\_c_1 + O(x)) + e^{\frac{1}{4x^3} - \frac{2}{3x}}(\_c_2 x^3 + O(x^4))\right]$$

8. We refine the coefficients of all terms of the equation:

```
> (4*x^4+O(x^9))*theta(y(x),x,2)+(3*x+O(x^6))*theta(y(x),x,1)+
(2+O(x^4))*y(x):
> FormalSolution(%,y(x));
```

$$\left[e^{\frac{2}{3x}}\left(\_c_1 - \frac{16\_c_1 x}{27} - \frac{196\_c_1 x^2}{729} + O(x^3)\right) + e^{\frac{1}{4x^3} - \frac{2}{3x}}\left(\_c_2 x^3 + \frac{16\_c_2 x^4}{27} + O(x^5)\right)\right]$$

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