D'Alembertian Series Solutions of LODE with Polynomial Coefficients

S.A. Abramov, Dorodnicyn Computing Centre of RAS, Moscow, Russia
M.A. Barkatou, XLIM UMR CNRS 6172, University of Limoges, France
sabramov@ccas.ru, moulay.barkatou@unilim.fr

Let E be the shift operator acting on sequences of complex numbers as $Ea_n = a_{n+1}$ for any sequence (a_n). The sequence a is d'Alembertian if for large enough values of the index n the elements a_n of the sequence satisfy a linear recurrence equation $R(a_n) = 0$, where

$$R = (E + f_1(n)) \circ \cdots \circ (E + f_k(n)), \quad f_i(n) \in \mathbb{C}(n).$$

Elements of a d'Alembertian sequence can be explicitly represented as a function of the index n using only rational functions, the gamma function and finite sums, e. g. the sequence $a_n = 2^n \sum_{k=0}^{n} \frac{(-1)^k}{\Gamma(k+1)}$ is d'Alembertian with $R = (E + \frac{2}{n+2}) \circ (E - 2)$. A d'Alembertian series is a formal power series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ whose coefficients sequence is d'Alembertian (this notion generalizes the notion of hypergemetric series, where the order k of the operator R is 1). Let L be a linear differential operator with polynomial coefficients, z_0 a fixed point in $\mathbb{C}$ and $\mathcal{A}_{z_0}$ the space of d'Alembertian series solutions of the equation $L(y) = 0$ at z_0. We prove that the dimension of $\mathcal{A}_{z_0}$ is the same for all ordinary (i.e., non-singular) points z_0 of L. In addition, we prove that if z_0 is an ordinary point of L then all d'Alembertian series solutions represent some analytic solutions which have a simple representation of the form

$$g_1(z) \int g_2(z) \int \cdots \int g_m(z)dz \ldots dzdz$$

where $g_i(z)$ is such that $\frac{g_i'(z)}{g_i(z)} \in \mathbb{C}(z)$. However the situation can be different if z_0 is a singular point.