D'Alembertian Series Solutions at Ordinary Points of LODE with Polynomial Coefficients

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Abstract

By definition, the coefficient sequence $\mathbf{c} = (c_n)$ of a d'Alembertian series — Taylor's or Laurent's — satisfies a linear recurrence equation with coefficients in $\mathbb{C}(n)$ and the corresponding recurrence operator can be factored into first order factors over $\mathbb{C}(n)$ (if this operator is of order 1, then the series is hypergeometric). Let L be a linear differential operator with polynomial coefficients. We prove that if the expansion of an analytic solution $u(z)$ of the equation $L(y) = 0$ at an ordinary (i.e., non-singular) point $z_0 \in \mathbb{C}$ of L is a d'Alembertian series, then the expansion of $u(z)$ is of the same type at *any* ordinary

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point. All such solutions are of a simple form. However the situation can be different at singular points.

1 Introduction

If one finds a finite number of coefficients of a power series solution of a differential equation at a fixed point, then this gives an approximate (or asymptotic) representation of this solution. If one finds a dependence of coefficients on values of the index n, and if this dependence can be described by some simple tools, e.g. as a function of n in a closed form, then one receives a full representation of the solution by an infinite series, though it may be that the solution itself as an analytic function has no closed form representation via elementary functions and quadratures. The opportunity of using such series for representing of differential equations solutions extends the notion of closed form solutions. A typical example is given by hypergeometric series. In this paper we consider more general type of d'Alembertian series which will be defined below.

Let E be the shift operator acting on sequences of complex numbers $\mathbf{c} = (c_n)$ as $E(\mathbf{c}) = \mathbf{b}$, where the sequence $\mathbf{b} = (b_n)$ is defined by the equality $b_n = c_{n+1}$.

Definition 1 *The sequence* $\mathbf{c}$ *is* d'Alembertian *if for large enough values of the index* n *the elements* c_n *of the sequence satisfy a linear recurrence equation* $M(\mathbf{c}) = 0$*, where*

$$M = (E - r_1(n)) \circ (E - r_2(n)) \circ \ldots \circ (E - r_m(n)), \qquad (1)$$

$r_1(n), r_2(n), \ldots, r_m(n) \in \mathbb{C}(n)$*. Any operator of the form (1) will be called* completely factorable*.*

Notice that any sequence with finite support (i.e., a sequence which has only finite set of non-zero elements) is d'Alembertian: we can take any completely factorable M in this case.

It is known that the elements (with large enough values of the index) of a d'Alembertian sequence can be explicitly represented by a function of the index n using only rational functions, the gamma function and finite sums

([1]), e.g. if $M = (E + \frac{1}{2(n+2)}) \circ (E - \frac{1}{2})$ then the equation $M(y) = 0$ is satisfied when $n \geq 0$ by two linearly independent sequences

$$2^{-n} \quad \text{and} \quad 2^{-n} \sum_{k=0}^{n} \frac{(-1)^k}{\Gamma(k+1)}.$$

Definition 2 *A power series $\sum_n c_n(z - z_0)^n$ is d'Alembertian if the sequence (c_n) is d'Alembertian (this notion generalizes the notion of hypergeometric series, where the order m of the operator (1) is 1).*

Let

$$L = \sum_{k=0}^{d} a_k(z) D^k \in \mathbb{C}[z, D], \tag{2}$$

$D = \frac{d}{dz} ='$. Assume that the leading coefficient $a_d(z)$ is a non-zero polynomial (so $\text{ord} L = d$), and that $a_0(z), \ldots, a_d(z)$ do not have a non-constant common factor. Recall that $z_0 \in \mathbb{C}$ is an *ordinary* point of L if $a_d(z_0) \neq 0$, otherwise z_0 is *singular*; this definition can be reformulated so as it will make sense when the coefficients of L are rational functions: z_0 is ordinary if the rational functions

$$\frac{a_0(z)}{a_d(z)}, \frac{a_1(z)}{a_d(z)}, \ldots, \frac{a_{d-1}(z)}{a_d(z)}$$

have no pole at z_0, otherwise z_0 is singular. If z_0 is an ordinary point of L then any formal power series $y = \sum_n c_n(z - z_0)^n$ satisfying $L(y) = 0$ is a convergent Taylor series, and the dimension of the $\mathbb{C}$-space of solutions of this type is $d = \text{ord} L$.

It has been shown in [2] that if at an ordinary point of L the expansion of a solution $u(z)$ of L is a hypergeometric series, then $u(z)$ has one of three possible forms:

$$f(z) + p(z)e^{vz}, \quad f(z) + p(z)(z - c)^w, \quad \frac{f(z)}{(z - c)^l} + p(z) \log(z - c),$$

where $f(z), p(z) \in \mathbb{C}[z]$, $v, w, c \in \mathbb{C}$, $l \in \mathbb{N}$. Furthermore any such a solution can be expanded into a hypergeometric series at any ordinary point of L.

In the present paper we generalize this result proving that if the expansion of an analytic solution of the equation $L(y) = 0$ at an ordinary point z_0 is a d'Alembertian series, then the expansion of this analytic solution is of the same type at any other ordinary point. As a consequence, the dimension of the space of d'Alembertian series solutions of $L(y) = 0$ is the same for all ordinary points of the operator L. We also prove that if $L(y) = 0$ has a d'Alembertian series solution at an ordinary point z_0 then it has also a solution of the form

$$\frac{\sum_{n=0}^{\infty} c_n (z - z_0)^n}{f(z)}, \tag{3}$$

where the numerator is a hypergeometric series and the denominator $f(z)$ is a polynomial (however a hypergeometric series solution does not exist in general; it might be that this looks quite surprising because it is well known that if a linear recurrence equation with polynomial coefficients has a d'Alembertian sequence solution, then it has a hypergeometric sequence solution as well). In addition, if z_0 is an ordinary point of L then all d'Alembertian series solutions at z_0 represent some analytic solutions which are of the form

$$g_1(z) \int g_2(z) \int \dots \int g_m(z)\, dz \dots dz dz,$$

with $m \leq \operatorname{ord} L$, and $g_i(z)$ is either of the form $r(z)e^{vz}$ or of the form $r(z)(z - c)^w$, with $r(z) \in \mathbb{C}(z) \setminus \{0\}$, $v, w, c \in \mathbb{C}$ (here $r(z), v, w, c$ depend on i, $i = 1, 2, \dots, m$).

It follows from the results of the present paper that solutions in the form of d'Alembertian series at ordinary points are of limited interest, since they represent quite simple functions, and, additionally, at each ordinary point we get d'Alembertian series expansion of the same solutions. So going from an ordinary point to another we get nothing new in this respect. As a contrast, the singular points of L can be of particular interest. However there is only a finite number of singular points, and one can check them using a step-by-step examination.

We also consider the point at infinity and, as it is usually done in the theory of linear ordinary differential equations, distinguish the cases of ordinary and singular point of L at infinity. It turns out that if the point at infinity is ordinary, then it is not improbable that there exists an analytic solution which has d'Alembertian series expansion at infinity while its Taylor expansion at any finite ordinary point is not a d'Alembertian series. Notice that up to Section 5 we consider only finite (i.e., belonging to $\mathbb{C}$) points.

In the rest of this paper L will always denote operator (2). For short, we will say about solutions of L instead of solutions of the equation $L(y) = 0$ and will use the same style in the recurrence operator case.

2 Preliminaries

We denote by $\mathbb{C}[z, z^{-1}, D]$ the non-commutative ring of polynomials in z, z^{-1} and D. The multiplication corresponds to the composition of operators and it is characterized by the following rules :

- $Dz = zD + 1$
- $Dz^{-1} = z^{-1}D - z^{-2}$
- the rings $\mathbb{C}[z, z^{-1}]$ and $\mathbb{C}[D]$ are commutative.

We denote by $\mathbb{C}[n, E, E^{-1}]$ the non-commutative ring of polynomials in n, E and E^{-1}. The multiplication corresponds to the composition of operators and it is characterized by the following rules :

- $En = (n + 1)E$
- $E^{-1}n = (n - 1)E^{-1}$
- the ring $\mathbb{C}[n]$ and $\mathbb{C}[E, E^{-1}]$ are commutative.

The correspondence $z \mapsto E^{-1}$, $D \mapsto (n + 1)E$, $z^{-1} \mapsto E$ defines an isomorphism $\mathcal{R}$ from $\mathbb{C}[z, z^{-1}, D]$ onto $\mathbb{C}[n, E, E^{-1}]$.

We will also consider the field

$$\mathbb{C}((z)) = \mathbb{C}[[z]][z^{-1}] \tag{4}$$

of power series of the form

$$\sum_{n=m}^{\infty} c_n z^n, \quad m \in \mathbb{Z}, \quad c_i \in \mathbb{C}. \tag{5}$$

The *coefficient sequence* of (5) is the double-sided sequence

$$\ldots, 0, 0, c_m, c_{m+1}, \ldots \tag{6}$$

(so we set $c_k = 0$ for all $k < m$).

It is well known that the application of L to (5) gives a series, whose coefficient sequence is the result of the application to (6) of the recurrence operator

$$R = \sum_{k=t}^{l} q_k(n)E^k \in \mathbb{C}[n, E, E^{-1}] \qquad (7)$$

which is the $\mathcal{R}$-image of L (see, e.g., [3]). We suppose that $q_l(n), q_t(n) \neq 0$ in (7) (note that it is possible that $t < 0$ and even $l < 0$).

For R of the form (7) we set $\mathrm{ord} R = l - t$.

In the sequel we will use some facts proven in [2, 3], the main of those facts can be formulated as in the following theorem.

Theorem 1 *Let 0 be an ordinary point of L and $R = \mathcal{R}L$. Suppose that R have no nonzero solution with finite support. If R is right divisible in $\mathbb{C}(n)[E, E^{-1}]$ by a first order monic operator $E - r(n)$, $r(n) \in \mathbb{C}(n) \setminus \{0\}$ then*

(i) the operator $E - r(n)$ has one of the forms:

$$E - v, \qquad (8)$$

$$E - \frac{v}{n+1}\frac{C(n+1)}{C(n)}, \qquad (9)$$

$$E - \frac{v}{n+1}\frac{C(n+1)}{C(n)}(n - w), \qquad (10)$$

where $v \in \mathbb{C} \setminus \{0\}$, $w \in \mathbb{C} \setminus \mathbb{N}$, $C(n) \in \mathbb{C}[n] \setminus \{0\}$;

(ii) according to the cases (8), (9), (10) the operator L either can be represented in the form $L' \circ (z^{-1} - v), L' \in \mathbb{C}[z, z^{-1}, D]$, or is right-divisible in $\mathbb{C}(z)[D]$ by a monic first-order operator of one of two forms:

$$D - \left(\frac{p'(z)}{p(z)} + v\right), \qquad (11)$$

$$D - \left(\frac{p'(z)}{p(z)} - \frac{vw}{1 - vz}\right), \qquad (12)$$

where $p(z) \in \mathbb{C}[z] \setminus \{0\}$ and, as in (i), $v \in \mathbb{C} \setminus \{0\}$, $w \in \mathbb{C} \setminus \mathbb{N}$.

Remark 1 *If L is right-divisible in $\mathbb{C}(z)[D]$ by (11) or by (12), then L has a solution*

$$p(z)e^{vz}, \tag{13}$$

or, resp.,

$$p(z)(1-vz)^w. \tag{14}$$

Solution (14) can be rewritten in the form

$$q(z)(z-c)^w,$$

$q(z) \in \mathbb{C}[z], c \in \mathbb{C} \setminus \{0\}$.

Remark 2 *Let 0 be an ordinary point of L and $R = \mathcal{R}L$. If R has a non-zero solution with finite support then L has a non-zero polynomial solution $p(z)$ and therefore is right-divisible in $\mathbb{C}(z)[D]$ by*

$$D - \frac{p'(z)}{p(z)}. \tag{15}$$

We will use also a well-known elementary fact on first order linear differential operators. Let $F \in \mathbb{C}[z, D]$, $\mathrm{ord} F = 1$. If $u(z)$ is a non-zero analytic solution of F then

$$\frac{u'(z)}{u(z)} \in \mathbb{C}(z),$$

and the general solution of an equation $F(y) = v(z)$ with analytic $v(z)$ is

$$u(z) \int \frac{v(z)}{u(z)} dz \tag{16}$$

(the integration constant can be taken arbitrary). If $L = \bar{L} \circ F$, $\bar{L} \in \mathbb{C}(z)[D]$, and $u_1, u_2, \ldots, u_d$ are linearly independent solutions (analytic functions or formal Laurent's series from $\mathbb{C}((z))$ of L such that $F(u_1) = 0$, then

$$u_1 \left(\frac{u_2}{u_1}\right)', u_1 \left(\frac{u_3}{u_1}\right)', \ldots, u_1 \left(\frac{u_d}{u_1}\right)' \tag{17}$$

are linearly independent solutions of $\bar{L}$.

3 Simple points

The statements of this paper are easier to prove, if we formulate them for a more general case than the case of an ordinary point.

Definition 3 *We call* $z_0 \in \mathbb{C}$ *a* simple *point of* L, *if there exists* $l \in \mathbb{N}$ *such that the function* $(z - z_0)^l u(z)$ *is holomorphic at* z_0 *(i.e. without singularities in a neighborhood of* z_0*) for any solution* $u(z)$ *of* L. *The minimal* l *with such a property will be called the* exponent *of* L *at* z_0 *(if the point* z_0 *is ordinary then* z_0 *is evidently a simple point, and the exponent of* L *at* z_0 *is* 0*).*

If 0 is a simple point of L then the exponent of L at 0 will be referred to as the exponent of L for short.

Remark 3 *Notice that if* 0 *is a simple point of* L *and the exponent of* L *is equal to* l *then, generally speaking,* 0 *is not an ordinary point of the operator* $L \circ z^{-l}$. *However if* 0 *is a singular point of* $L \circ z^{-l}$, *then* 0 *is an apparent singularity, and there exists an operator, which, first, is right-divisible in* $\mathbb{C}(z)[D]$ *by* $L \circ z^{-l}$, *and, second,* 0 *is an ordinary point of this operator ([5], [4]). We will denote by* $L^{\smile}$ *an arbitrary operator having such properties. If* 0 *is an ordinary point of* $L \circ z^{-l}$, *then we can set* $L^{\smile} = L \circ z^{-l}$.

We will denote the set of d'Alembertian series of the form $\sum_{n=0}^{\infty} c_n z^n$ by Ser_A. The notation $\mathrm{Ser}_A(L)$ will be used for the set of solutions of L belonging to Ser_A. The sets Ser_A, $\mathrm{Ser}_A(L)$ are $\mathbb{C}$-linear spaces [1]. We can extend the notions of hypergeometric and, resp., d'Alembertian series, considering in addition Laurent series with hypergeometric and d'Alembertian coefficient sequences. The corresponding $\mathbb{C}$-linear space of d'Alembertian Laurent series will be denoted by Ser_A^-. One has $\mathrm{Ser}_A \subset \mathrm{Ser}_A^-$. We will also consider the corresponding solution space $\mathrm{Ser}_A^-(L)$.

Remark 4 *Suppose that* 0 *is a simple point of* L *and let* $\varphi(z) \in \mathbb{C}((z))$ *be a formal solution of* L. *Then* $\varphi(z)$ *is a convergent series (in a punctured neighborhood of* 0*) and its sum* $\Phi(z)$ *is an analytic solution of* L. *This follows directly from Definition 3.*

Lemma 1 *Let* $f(z) \in \mathbb{C}[z] \setminus \{0\}$ *be an arbitrary nonzero polynomial. Then* *(i)* *if* 0 *is a simple point of* L *then it is a simple point of* $L \circ f(z)$ *and* $L \circ [f(z)]^{-1}$*;*

(ii) $\dim \mathrm{Ser}_A^-(L) = \dim \mathrm{Ser}_A^-(L \circ f(z)) = \dim \mathrm{Ser}_A^-(L \circ [f(z)]^{-1})$.

Proof. We note first that it suffices to prove the lemma for $f(z) = z - c$, with $c \in \mathbb{C}$.

(i): If $c \neq 0$ then the exponent of $L \circ (z-c)$ does not exceed the exponent l of L. The exponent of $L \circ z$ does not exceed $l+1$. The exponent of $L \circ (z-c)^{-1}$ does not exceed l.

(ii): Let $\varphi(z) \in \mathrm{Ser}_A^-$, then the coefficient sequence of the series $(z-c)\varphi(z)$ is the result of action of the linear difference operator $P = E^{-1} - c$ on the d'Alembertian coefficient sequence of $\varphi(z)$. The result is a d'Alembertian sequence ([1]). If the coefficient sequence of $\varphi(z)$ is a solution of a completely factorable difference operator M, then the coefficient sequence of $\varphi(z)/(z-c)$ is a solution of the operator $M \circ P$. This implies that the coefficient sequence of $\varphi(z)/(z-c)$ is a solution of a completely factorable operator and therefore is d'Alembertian. So the multiplication by $(z-c)$ and the multiplication by $(z-c)^{-1}$ can be viewed as linear maps from Ser_A^- into itself. The kernel of each of these linear maps is zero. Therefore, the image of a finite-dimensional subspace of Ser_A^- (e.g., the space $\mathrm{Ser}_A^-(L)$) by each of this transformations, is a subspace of the same dimension. □

The following proposition is a consequence of Lemma 1.

Proposition 1 *Let* 0 *be a simple point of* L *and* $r(z) \in \mathbb{C}(z) \setminus \{0\}$*. In this case*

(i) 0 *is a simple point of* $L \circ r(z)$*,*

(ii) the multiplication by $r(z)$ *is a linear transformation of* Ser_A^- *onto* Ser_A^- *with zero kernel;*

(iii) $\dim \mathrm{Ser}_A^-(L) = \dim \mathrm{Ser}_A^-(L \circ r(z))$.

If W is a subset of $\mathbb{C}((z))$ consisting of convergent Laurent series (see Remark 4), then we will denote by $\langle W \rangle$ the set of all analytic functions with a Laurent series expansion (at 0) belonging to W.

Lemma 2 *Let* 0 *be a simple point of* L *and suppose that* $L = L' \circ G$*, where* $L' \in \mathbb{C}(z)[D]$ *and* G *is a first order operator of the form (11), (12) or (15) with* $p(z) = 1$*. In this case*

(i) 0 *is a simple point of* L'*;*

(ii) $\mathrm{Ser}_A^-(L) \neq 0$*, and* $\dim \mathrm{Ser}_A^-(L) = \dim \mathrm{Ser}_A^-(L') + 1$*;*

(iii) if $\Phi(z)$ *is a non-zero analytic solution of* G*, then the set of analytic functions representable by series belonging to* $\mathrm{Ser}_A^-(L)$ *is*

$$\langle \mathrm{Ser}_A^-(L) \rangle = \Phi(z) \int \frac{\langle \mathrm{Ser}_A^-(L') \rangle}{\Phi(z)} dz, \tag{18}$$

i.e., the set of all functions of the form

$$\Phi(z) \int \frac{\Psi(z)}{\Phi(z)} dz, \quad \Psi(z) \in \langle \mathrm{Ser}_A^-(L') \rangle.$$

Proof.

(i): Let $\psi_1(z), \psi_2(z), \ldots, \psi_d(z) \in \mathbb{C}((z))$ be linearly independent solutions of L such that $G(\psi_1(z)) = 0$. Then by formula (17) the Laurent's series

$$\psi_1(z) \left(\frac{\psi_2(z)}{\psi_1(z)} \right)', \psi_1(z) \left(\frac{\psi_3(z)}{\psi_1(z)} \right)', \ldots, \psi_1(z) \left(\frac{\psi_d(z)}{\psi_1(z)} \right)'$$

are linearly independent solutions of L' belonging to $\mathbb{C}((z))$.

(ii): First notice that if G is of the form (12) with $p(z) = 1$, then we can rewrite any equation $G(y) = \psi(z)$, $\psi(z) \in \mathbb{C}((z))$, as $\bar{G}(y) = \bar{\psi}(z)$, where $\bar{G} = (1 - vz)D + vw, \bar{\psi}(z) = (1 - vz)\psi(z)$. By Lemma 1 $\psi(z) \in \mathrm{Ser}_A^-$ implies $(1 - vz)\psi(z) \in \mathrm{Ser}_A^-$. Moreover if $L = \bar{L}' \circ \bar{G}$, then 0 is a simple point of $\bar{L}'$. For saving the old notation, we will assume that G has one of two following forms:

$$D - v, \quad (1 - vz)D + vw, \;\; v \in \mathbb{C}, \; w \in \mathbb{C} \setminus \mathbb{N}. \tag{19}$$

Set $R_G = \mathcal{R}G$. Evidently $\mathrm{ord} R_G = 1$.

The rest of the proof of (ii) will be divided into a few short steps.

a) If $\varphi(z) \in \mathrm{Ser}_A^-$ then $G(\varphi(z)) \in \mathrm{Ser}_A^-$ since the coefficient sequence of $G(\varphi(z))$ is obtained by applying the first order difference operator R_G to the coefficient sequence of $\varphi(z)$.

b) If $G(\varphi(z)) = \psi(z)$ where $\psi(z) \in \mathrm{Ser}_A^-$ and $\varphi(z) \in \mathbb{C}((z))$, then $\varphi(z) \in \mathrm{Ser}_A^-$. Indeed, let M be a completely factorable operator which annihilates the coefficient sequence of $\psi(z)$. Then $M \circ R_G$ annihilates the coefficient sequence of $\varphi(z)$. But $M \circ R_G$ is a completely factorable, since R_G is a first order operator. Therefore $\varphi(z) \in \mathrm{Ser}_A^-$.

c) The inequality $\dim \mathrm{Ser}_A^-(L) \leq \dim \mathrm{Ser}_A^-(L') + 1$ is valid since, first, $\dim \mathrm{Ser}_A^-(G) = 1$, and, second, by a) we have $G(\varphi(z)) \in \mathrm{Ser}_A^-(L')$ for any $\varphi(z) \in \mathrm{Ser}_A^-(L)$.

d) Now to complete the proof of (ii) it is sufficient to show that for any $\psi(z) \in \mathrm{Ser}_A^-(L')$ there exists $\varphi(z) \in \mathrm{Ser}_A^-$ such that $G(\varphi(z)) = \psi(z)$.

   Let $\Phi(z)$ be a non-zero analytic solution of G. This function can be taken in one of two forms: e^{vz}, or $(1 - vz)^w$, $v \in \mathbb{C}$, $w \in \mathbb{C} \setminus \mathbb{N}$. In addition, let $\Psi(z)$ be a function that is represented by the series $\psi(z)$ (see (i) and Remark 4). Then by formula (16) the equation $G(y) = \Psi(z)$ has the analytic solution

$$\Phi(z) \int \frac{\Psi(z)}{\Phi(z)} dz \tag{20}$$

   (one can take any fixed integration constant). This function is an analytic solution of L, and must be meromorphic since 0 is a simple point of L. Therefore the series that represents this function belongs to $\mathbb{C}((z))$. By b) this series belongs to Ser_A^-.

(iii): Follows from (16).

□

Proposition 2 *Let* 0 *be a simple point of* L *and* $\mathrm{Ser}_A^-(L) \neq 0$. *In this case any element of* $\mathrm{Ser}_A^-(L)$ *represents a function of the form*

$$h_1(z) \int h_2(z) \int \dots \int h_m(z)\, dz \dots dz dz, \tag{21}$$

with $m \leq \mathrm{ord} L$, *and* $h_i(z)$ *is either of the form* $r(z)e^{vz}$ *or of the form* $r(z)(1-vz)^w$, *with* $r(z) \in \mathbb{C}(z) \setminus \{0\}$, $v, w \in \mathbb{C}$ *(where* $r(z), v, w$ *depend on* i*)*, $i = 1, 2, \dots, m$.

Proof. Follows from (18). □

Definition 4 *We call an h-*factor *a differential operator H of the form*

$$F \circ r(z), \tag{22}$$

where $r(z) \in \mathbb{C}(z) \setminus \{0\}$, and F is a first order operator of the form (11), (12) or (15).

Proposition 3 *Let 0 be a simple point of L and $L = L' \circ H$, where $L' \in \mathbb{C}(z)[D]$, and H is an h-factor. In this case*

(i) 0 *is a simple point of L',*

(ii) $\mathrm{Ser}_A^-(L) \neq 0$, *and* $\dim \mathrm{Ser}_A^-(L) = \dim \mathrm{Ser}_A^-(L') + 1$,

(iii) $\langle \mathrm{Ser}_A^-(L) \rangle$ *is described by formula (18), where Φ is a non-zero analytic solution of H.*

Proof. Let H be of the form (22), and $p(z)$ be a polynomial involved into F as in formulas (11), (12), (15). We have $L = L' \circ F \circ p(z) \circ r_1(z)$, where $r_1(z) = [p(z)]^{-1} r(z) \in \mathbb{C}(z)$, and $L = L' \circ p(z) \circ G \circ r_1(z)$, where G is represented by one of formulas (11), (12), (15) with $p(z) = 1$. It is easy to see that

a) 0 is a simple point of $L' \circ p(z) \circ G$ and $\dim \mathrm{Ser}_A^-(L' \circ p(z) \circ G) = \dim \mathrm{Ser}_A^-(L)$ by Proposition 1;

b) 0 is a simple point of $L' \circ p(z)$ and $\dim \mathrm{Ser}_A^-(L' \circ p(z)) + 1 = \dim \mathrm{Ser}_A^-(L' \circ p(z) \circ G) = \dim \mathrm{Ser}_A^-(L)$ by a) and Lemma 2;

c) 0 is a simple point of L' and $\dim \mathrm{Ser}_A^-(L') = \dim \mathrm{Ser}_A^-(L' \circ p(z))$ by b) and Proposition 1.

The claimed in (i), (ii) follows. The proof of (iii) is the same as in Lemma 2, i.e., by formula (16). □

Proposition 4 *Let 0 be a simple point of L, and $\mathrm{Ser}_A^-(L) \neq 0$. In this case* *(i) L is right divisible in $\mathbb{C}(z)[D]$ by an operator of the form*

$$F \circ f(z), \tag{23}$$

where $f(z) \in \mathbb{C}[z]$, and F is a first order operator of the form (11), (12) or (15),

(ii) L has a solution of the form

$$\frac{\sum_{n=0}^{\infty} c_n z^n}{f(z)}, \tag{24}$$

where the power series $\sum_{n=0}^{\infty} c_n z^n$ has a hypergeometric coefficient sequence and $f(z) \in \mathbb{C}[z]$.

Proof.

(i): We can represent L in the right-coefficient form:

$$L = \sum_{k=0}^{d} D^k \circ b_k(z), \quad b_0(z), b_1(z), \ldots, b_d(z) \in \mathbb{C}[z]. \tag{25}$$

Let $f(z) = \gcd(b_0(z), b_1(z), \ldots, b_d(z))$, $L = L' \circ f(z)$, $L' \in \mathbb{C}[z, D]$. Then 0 is a simple point of L', since the leading coefficient of L' divides $b_d(z) = a_d(z)$. It is sufficient to consider the case $\deg f(z) = 0$: by Lemma 1(i) we have $\mathrm{Ser}_A^-(L') \neq 0$ and if L' has a right divisor of the form (23) then evidently L has a right divisor of such a form too. In the rest of the proof we suppose that $\deg f(z) = 0$.

Set $R = \mathcal{R}L, R_1 = \mathcal{R}L^{\smile}$ (see Remark 3 for the definition of $L^{\smile}$). If R has a nonzero solution with finite support then L has a right divisor of the form (15) and there is nothing to prove. Suppose R has no such a solution. Then the operators R, R_1 have in $\mathbb{C}(n)[E, E^{-1}]$ a common right divisor M of the form (1) (let $\mathbf{c} = (c_n)$ be the coefficient sequence of a nonzero element of $\mathrm{Ser}_A^-(L)$; we can take a completely factorable operator M of minimal order such that $M(\mathbf{c}) = 0$ for all large enough values of the index n. Therefore R and R_1 have a common right divisor of the form $E - r(n)$, $r(n) \in \mathbb{C}(n) \setminus \{0\}$. We claim that $r(n) \notin \mathbb{C}$. Indeed, otherwise $r(n) = c \in \mathbb{C} \setminus \{0\}$, and L is right divisible in $\mathbb{C}[z, z^{-1}, D]$ by $z^{-1} - c$; this implies that $L = L' \circ (z - \frac{1}{c})$, where $L' \in \mathbb{C}[z, D]$ because $L \in \mathbb{C}[z, D]$. This contradicts the condition $\deg f(z) = 0$. So $r(n) \notin \mathbb{C}$. Since 0 is an ordinary point of $L^{\smile}$ the operator

$E - r(n)$ has one of the forms (9), (10) by Theorem 1(i). By Theorem 1(ii) L has a right divisor of one of the forms (11), (12). The claim follows.

(ii): The statement follows from (i), since F has a series solution with a hypergeometric coefficient sequence. □

Example 1 *Consider the operator* $L = (z-1)D-(z-2)$. *The space* $\mathrm{Ser}_A(L)$ *is generated by the series*

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{1}{k!}\right)z^n.$$

This series is equal to

$$\frac{\sum_{n=0}^{\infty}\frac{1}{n!}z^n}{1-z},$$

where the numerator is a hypergeometric series, and the denominator is a polynomial. The operator L *has no hypergeometric series solution. Notice that* $L = D \circ (z-1) - (z-1) = (D-1) \circ (z-1)$.

Remark 5 *It follows from the proof of Proposition 4(i) that if* 0 *is a simple point of* L, $\mathrm{Ser}_A(L) \neq 0$, *and* $b_0(z), b_1(z), \dots, b_d(z)$ *in (25) have no common root, then* $\mathrm{Ser}_A^-(L)$ *contains a hypergeometric series solution, which is the expansion of an analytic solution of one of the forms*

$$p(z)e^{vz}, \;\; p(z)(c-z)^w, \tag{26}$$

$p(z) \in \mathbb{C}[z] \setminus \{0\}$, $v, w \in \mathbb{C}, c \in \mathbb{C} \setminus \{0\}$ *(we use Remark 1).*

As a consequence of Propositions 2, 3, 4 we have the following

Theorem 2 *Let* 0 *be a simple point of* L. *In this case*

(i) if $\mathrm{Ser}_A^-(L) \neq 0$, *then* L *is right divisible in* $\mathbb{C}(z)[D]$ *by an h-factor;*

(ii) if $L = L' \circ H$, *where* $L' \in \mathbb{C}(z)[D]$ *and* H *is an h-factor, then*

0 *is a simple point of* L',

$\mathrm{Ser}_A^-(L) \neq 0$, *and* $\dim \mathrm{Ser}_A^-(L) = \dim \mathrm{Ser}_A^-(L') + 1$,

formula (18) is valid where $\Phi(z)$ *is a non-zero analytic solution of* H,

L has a solution of the form (24);

(iii) if $L = L' \circ H_m \circ \ldots \circ H_2 \circ H_1$, where an operator $L' \in \mathbb{C}(z)[D]$ is not right-divisible by any h-factor, and $H_1, H_2, \ldots, H_m$ are h-factors, then

$\dim \mathrm{Ser}_A^-(L) = m$,

$\langle \mathrm{Ser}_A^-(L) \rangle$ is the space of analytic solutions of the operator $H_m \circ \ldots \circ H_2 \circ H_1$,

any element of $\langle \mathrm{Ser}_A^-(L) \rangle$ is an analytic function of the form (21).

As a consequence of Theorem 2 we have $\mathrm{Ser}_A^-(L) \neq 0$ iff L is right divisible by an h-factor.

4 Space of d'Alembertian series solutions at an arbitrary simple point

The aim of this section is an investigation of the spaces of d'Alembertian series solutions of L at different simple points. We will exploit the fact that the operator L of the form (2) has a solution $\sum_n c_n(z - z_0)^n$ iff the operator

$$L_{z+z_0} = \sum_{k=0}^{d} a_k(z + z_0)D^k$$

has the solution $\sum_n c_n z^n$. It is clear that point $z_0 \in \mathbb{C}$ is a simple point of L iff 0 is a simple point of L_{z+z_0}.

Proposition 5 *Starting from L one can construct an operator $\widetilde{L} \in \mathbb{C}(z)[D]$ of order m, $0 \leq m \leq \mathrm{ord}L$, which has the form $\widetilde{L} = H_m \circ \ldots \circ H_2 \circ H_1$, where $H_i = G_i \circ r_i(z)$, $r_i(z) \in \mathbb{C}(z) \setminus \{0\}$, G_i is a monic first-order operator of one of two forms:*

$$D - v, \tag{27}$$

$$D + \frac{vw}{1 - vz} \tag{28}$$

with $v \in \mathbb{C}$, $w \in \mathbb{C} \setminus \mathbb{N}$, $i = 1, 2, \ldots, m$. Moreover, for any simple point z_0 of L one has

(i) $\dim \mathrm{Ser}_A^-(L_{z+z_0}) = \mathrm{ord}\widetilde{L}$;

(ii) $\mathrm{Ser}^-_A(L_{z+z_0}) = \mathrm{Ser}^-_A((\widetilde{L})_{z+z_0})$*;*

(iii) $\langle \mathrm{Ser}^-_A(L_{z+z_0}) \rangle$ *is the space of analytic solutions of* $(\widetilde{L})_{z+z_0}$*.*

Proof. We may suppose that 0 is a simple point of L, otherwise we could pick any simple (e.g., ordinary) point $z' \in \mathbb{C}$ of L and consider $L_{z+z'}$ instead of L. Then it follows from Theorem 2 that such $\widetilde{L}$ can be easily constructed if we consider $z_0 = 0$. Take such $\widetilde{L}$ and show that it satisfies the claimed conditions. Let L' be such that $L = L' \circ \widetilde{L}$. The operator L' has no right divisor in the form of an h-factor. Now let z_0 be an arbitrary simple point. We have $L_{z+z_0} = (L')_{z+z_0} \circ (\widetilde{L})_{z+z_0}$. Notice that the operator $(L')_{z+z_0}$ has no right divisor in the form of an h-factor, and $(\widetilde{L})_{z+z_0} = (H_m)_{z+z_0} \circ \ldots \circ (H_2)_{z+z_0} \circ (H_1)_{z+z_0}$ (i.e., m is independent on z_0). If H_i, $1 \leq i \leq m$, is of the form (27), then $(H_i)_{z+z_0}$ is of the form (27) as well. Suppose that H_i, $1 \leq i \leq m$, is of the form (28), then $(H_i)_{z+z_0}$ is of the form (28) iff $vz_0 \neq 1$. However $(z-z_0)^{-l}(z-z_0)^w$ is holomorphic at z_0 for some $l \in \mathbb{N}$ iff $w \in \mathbb{Z}$. In this case rewrite $(G_i)_{z+z_0}$ in the form $z^{-w} D \circ z^w$. If $i = m$ then joint the factor z^{-w} with $(\widetilde{L})_{z+z_0}$, otherwise joint it with $(r_{i+1}(z))_{z+z_0}$, i.e., with $r_{i+1}(z+z_0)$. So we can suppose that $(H_i)_{z+z_0}$ is of the form (28). The claimed follows from Theorem 2. □

The main result of this paper is

Theorem 3 *Let* z_0 *be a simple (in particular, an ordinary) point of* L*. In this case*

(i) if the expansion of an analytic solution $u(z)$ *of* L *is a d'Alembertian series at* z_0*, then the expansion of* $u(z)$ *is of the same type at any simple point;*

(ii) if the expansion of an analytic solution $u(z)$ *of* L *is a d'Alembertian series at* z_0*, then* $u(z)$ *is of the form*

$$g_1(z) \int g_2(z) \int \ldots \int g_m(z)\, dz \ldots dz dz, \tag{29}$$

with $m \leq \mathrm{ord} L$*, and* $g_i(z)$ *is either of the form* $r(z)e^{vz}$ *or of the form* $r(z)(z-c)^w$*, with* $r(z) \in \mathbb{C}(z) \setminus \{0\}$*,* $v, w, c \in \mathbb{C}$ *(where* $r(z), v, w, c$ *depend on* i*),* $i = 1, 2, \ldots, m$*;*

(iii) if L has a non-zero d'Alembertian series solution

$$\sum_{n=k}^{\infty} c_n (z - z_0)^n, \tag{30}$$

then L has a solution of the form

$$\frac{\sum_{n=0}^{\infty} c'_n (z - z_0)^n}{f(z)},$$

where the numerator is a non-zero hypergeometric series and the denominator is a polynomial;

(iv) if L has a non-zero d'Alembertian series solution (30) and the right-coefficient form (25) of L is such that $b_0(z), b_1(z), \ldots, b_d(z)$ have no common root, then L has a non-zero solution in the form of a hypergeometric series $\sum_{n=0}^{\infty} c_n z^n$ which represents a function of one of two forms (26).

Proof. Let $\widetilde{L}$ be the operator which corresponds to L as described in Proposition 5.

(i): By Proposition 5 (iii), since $u(z)$ is a solution of $\widetilde{L}$.

(ii): We can substitute $z + z_0$ for z into (21) and after an easy transformation receive (29) (the rational functions $r_i(z)$ can be changed).

(iii): By Proposition 5 and Theorem 2.

(iv): If z_0 is an ordinary point then the statement follows from Remark 5. If the point z_0 is simple but not ordinary then pick an ordinary point z_1. By (i) $\mathrm{Ser}^-_A(L_{z+z_0}) \neq 0$ implies $\mathrm{Ser}^-_A(L_{z+z_1})$. The coefficients of the right-coefficient form of L_{z+z_1} evidently have no common root. By Remark 5 L_{z+z_1} has a solution of one of two forms (26), and as a consequence L_{z+z_0} has a solution of this form (w.l.g. we can assume that $z_1 = 0$). It is clear that $p(z)e^{vz}$ has a hypergeometric expansion at z_0. Since z_0 is a simple point, the solution $p(z)(z - c)^w$ must have at z_0 either no singularity or a pole (c can be equal to 0). In both cases the expansion of this solution at z_0 is a hypergeometric series. □

The following two examples show that the situation at a singular point can differ from one at ordinary points.

Example 2 *Consider the operator*

$$L = \left(z^2 + z - 2\right) D^2 + \left(z^2 - z\right) D - \left(6\, z^2 + 7\, z\right).$$

L has two singular points, $z = 1$ and $z = -2$. The point $z = 1$ is not a simple point (since local solutions of L at $z = 1$ contains a logarithm term) while $z = -2$ is a simple point of L. A basis of solutions of L is

$$(z-1)\, e^{2\,z}, \quad (z-1)\, e^{2\,z} \int \left(\frac{z+2}{z-1}\right)^2 e^{5\,z} dz.$$

Notice that $\dim \operatorname{Ser}_A^-(L) = \dim \operatorname{Ser}_A^-(L_{z+z_0}) = 2$ *for all* $z_0 \in \mathbb{C} \setminus \{1\}$ *while* $\dim \operatorname{Ser}_A^-(L_{z+z_0}) = 1$ *if* $z_0 = 1$. *The right-coefficients form of L is*

$$L = D^2 \left(z^2 + z - 2\right) + D \left(z^2 - 5\, z - 2\right) - \left(6\, z^2 + 9\, z - 3\right).$$

The right-coefficients of L have no common root and L has a hypergeometric series solution as expected. Notice also that L can be factorized as $L' \circ F$:

$$L' = \left(\left(z^2 + z - 2\right) D + 3\, z^2 + 2\, z - 2\right) \circ (z-1),$$

$$F = (D-2) \circ \frac{1}{z-1}.$$

This example shows that the dimension of the space of d'Alembertian series solutions at a singular point may be less than the dimension of the space of d'Alembertian series solutions at an ordinary point.

Example 3 *Consider the operator*

$$L = 9\, zD^2 + 6\, D - 1.$$

The only singular point is $z = 0$, which is not a simple point (the roots of the indicial equation at 0 are $\frac{1}{3}$ and 0).

One can verifies that L is irreducible over $\mathbb{C}(z)$ (check it by using the command DEtools[DFactor](L, [D, z]) of Maple). Hence $\dim \operatorname{Ser}_A^-(L) = \dim \operatorname{Ser}_A^-(L_{z+z_0}) = 0$ *for all* $z_0 \in \mathbb{C} \setminus \{0\}$.

If $z_0 = 0$, then the space $\operatorname{Ser}_A^-(L_{z+z_0})$ *is generated by the hypergeometric series*

$$\sum_{n=0}^{\infty} \frac{z^n}{9^n \Gamma\left(n + 2/3\right) \Gamma\left(n+1\right)}.$$

Note that the space of all local solutions of L at $z = 0$ is generated by the above series and the Frobenius series

$$z^{1/3} \sum_{n=0}^{\infty} \frac{z^n}{9^n \Gamma(n + 4/3) \Gamma(n + 1)}.$$

This example shows that the dimension of the space of d'Alembertian series solutions at a singular point may exceed the dimension of the space of d'Alembertian series solutions at an ordinary point.

5 The point at infinity

Definition 5 *The point $z = \infty$ is an* ordinary (simple) *point of L, if the point $t = 0$ is an ordinary (simple) point of the operator $\check{L}$, which can be constructed by the substitution*

$$z = \frac{1}{t}, \quad D = -t^2 D_t \tag{31}$$

into L (here $D_t = \frac{d}{dt}$). If the point ∞ is simple, then each non-zero solution of $\check{L}$ can be expanded into a series

$$b_k t^k + b_{k+1} t^{k+1} + \dots, \quad k \in \mathbb{Z} \tag{32}$$

with coefficients from $\mathbb{C}$. Each such a series gives the solution

$$b_k z^{-k} + b_{k+1} z^{-k-1} + \dots \tag{33}$$

of L. The series (33) is d'Alembertian *if the series (32) is d'Alembertian in the sense of Definition 2.*

In Lemma 3 and Theorem 4 below $\widetilde{L}$ is the operator which corresponds to L as described in Proposition 5.

Lemma 3 *Let ∞ be a simple point of L, and* $\operatorname{ord}\widetilde{L} = m > 0$. *Then*

$$\widetilde{L} = D \circ r_m(z) \circ \dots \circ D \circ r_2(z) \circ D \circ r_1(z), \tag{34}$$

where $r_i(z) \in \mathbb{C}(z)$, $i = 1, 2, \dots, m$.

Proof. It follows from Proposition 5. Indeed, using the notation of Proposition 5, if for i, $1 \leq i \leq m$, either

$$H_i = D - v, \quad v \neq 0,$$

or

$$H_i = D - \frac{vw}{1 - vz}, \quad v \neq 0, w \notin \mathbb{Z},$$

then ∞ is not evidently a simple point. □

Theorem 4 *Let $z_0 \in \mathbb{C}$ and ∞ be simple (in particular, ordinary) points of L. Let $u(z)$ be an analytic solution of L and suppose that its expansion at z_0 is a d'Alembertian series. In this case*

(i) at any simple point (including ∞) of L the series expansion of $u(z)$ is a d'Alembertian series;

(ii) $u(z)$ can be represented in the form (21) with $h_i(z) \in \mathbb{C}(z)$, $i = 1, 2, \ldots, m$;

(iii) L has a solution in $\mathbb{C}(z)$.

Proof.

(i): It follows from Lemma 3, Theorem 3, and the fact that the substitution (31) transforms (34) into an operator of the form

$$s_{m+1}(t)D_t \circ s_m(t) \circ \cdots \circ s_2(t)D_t \circ s_1(t),$$

$s_1(t), s_2(t), \ldots, s_{m+1}(t) \in \mathbb{C}(t)$.

(ii), (iii): By (i). □

Example 4 *Consider the first order operator $L = z^2D+1$ which has $ce^{\frac{1}{z}}, c \in \mathbb{C}$, as the general solution. Here the point at infinity is an ordinary point and the expansion series of $e^{\frac{1}{z}}$ at infinity is the hypergeometric series*

$$\sum_{n=0}^{\infty} \frac{1}{n!} z^{-n}.$$

However L has no nonzero d'Alembertian series solution at any finite point.

The example shows that one may get nonzero d'Alembertian series solutions at ∞ even if such solutions do not exist at simple points belonging to $\mathbb{C}$.

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References

[1] Abramov S.A., Petkovšek M. D'Alembertian Solutions of Linear Operator equations. *Proc. ISSAC'94*, Oxford (2004), 169–174.

[2] Abramov S.A., Petkovšek M., Special Power Series Solutions of Linear Differential Equations, in: D. Stanton (Ed.), Proc. FPSAC'96, Univ. of Minnesota (1996), 1–8.

[3] Abramov S.A., Petkovšek M., Ryabenko A. Special Formal Solutions of Linear Operator Equations. *Discrete Mathematics* (2000) **210**, 3–25.

[4] Abramov S.A., Barkatou M.A., van Hoeij M. Apparent Singularities of Linear Difference Equations with Polynomial Coefficients. *Applicable Algebra in Engeneering, Communications and Computing* (2006), **17**, N 2, 117–133.

[5] Tsai H. Weyl Closure of Linear Differential Operator, *J. Symb. Comput.* (2000) **29**, 747–775.