Computable Infinite Power Series in the Role of Coefficients of Linear Differential Systems

S.A. Abramov[1,⋆] and M.A. Barkatou[2]

[1] Computing Centre of the Russian Academy of Sciences, Vavilova, 40, Moscow 119333, Russia
sergeyabramov@mail.ru

[2] Institut XLIM, Département Mathématiques et Informatique, Université de Limoges; CNRS, 123, Av. A. Thomas, 87060 Limoges cedex, France
moulay.barkatou@unilim.fr

Abstract. We consider linear ordinary differential systems over a differential field of characteristic 0. We prove that testing unimodularity and computing the dimension of the solution space of an arbitrary system can be done algorithmically if and only if the zero testing problem in the ground differential field is algorithmically decidable. Moreover, we consider full-rank systems whose coefficients are computable power series and we show that, despite the fact that such a system has a basis of formal exponential-logarithmic solutions involving only computable series, there is no algorithm to construct such a basis.

1 Introduction

Linear ordinary differential systems with variable coefficients appear in various areas of mathematics. Power series are very important objects in the representation of the solutions of such systems as well as of the systems themselves. The representation of infinite series lies at the core of computer algebra. A general formula that expresses the coefficients of a series is not always available and may even not exist. One natural way to represent the series is the algorithmic one, i.e., providing an algorithm which computes its coefficients. Such algorithmic representation of a concrete series is not, of course, unique. This non-uniqueness is one of the reasons for undecidability of the zero testing problem for such computable series.

At first glance, it may seem that if we cannot decide algorithmically whether a concrete coefficient of a system is zero or not, then we will not be able to solve any more or less interesting problem related to the search of solutions. However, this is not completely right: at least, if we know in advance that the system is of full rank then some of the problems can still be solved. For example, we can find

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Laurent series [3] and regular [6] solutions. Some non-trivial characteristics can be computed as well, e.g., the so called "width" of the system [3]. Nevertheless, many of the problems are undecidable. For example, we cannot answer algorithmically the following question: does a given full-rank system with power series coefficients have a formal exponential-logarithmic solution which is not regular? We prove this undecidability in the present paper. It is also shown that if exponential-logarithmic solutions of a given full-rank system exist then there exists a basis of the space of those solutions such that all the series which appear in the elements of the basis are computable; the exact formulation is given in Proposition 7 of this paper.

So, we know that there exists a basis of the solution space which consists of computable objects, but we are not able to find this basis algorithmically. This is analogous to some facts of constructive mathematical analysis. In fact, the notion of a constructive real number (computable point) is fundamental in that discipline: "... an algorithm which finds the zeros of any alternating, continuous, computable function is impossible. At the same time, there cannot be a computable function that assumes values of different signs at the ends of a given interval and does not vanish at any computable point of this interval (a priori, it is impossible to rule out the existence of computable alternating functions whose zeros are all 'noncomputable'). These results are due to Tseitin [21] ..." ([14, p. 5], see also [16, §24]).

We prove in the same direction that testing unimodularity, i.e., the invertibility of the corresponding operator and computing the dimension of the solution space of an arbitrary system can be done algorithmically if and only if the zero testing problem in the ground differential field is algorithmically decidable. As a consequence, these problems are undecidable when the coefficients are power series or Laurent series which are represented by arbitrary algorithms.

If the algorithmic way of series representation is used then some of the problems related to linear ordinary systems are decidable while others are not. Note that the above mentioned algorithms for finding Laurent series solutions and regular solutions are implemented in Maple [23]. The implementation is described in [3,6] and, is available at http://www.ccas.ru/ca/doku.php/eg.

The rest of the paper is organized as follows: After stating some preliminaries in Section 2, we give in Section 3 a review of some results related to systems whose coefficients belong to a field K of characteristic zero. The field K is supposed to be a constructive differential field, i.e., there exist algorithms for the field operations, differentiation, and for zero testing. The problems that are listed in Section 3 can be solved algorithmically. On the other hand, we show in Section 4 that the same problems are algorithmically undecidable, if the field K is semi-constructive, i.e., there exist algorithms for the field operations and differentiation but there is no algorithm for zero testing. Finally, we consider in Section 5 semi-constructive fields of computable formal Laurent series in the role of coefficient field of systems of linear ordinary differential systems.

The results of this paper supplement known results on the zero testing problem and some algorithmically undecidable problems related to differential equations (see, e.g., [10], [13]).

2 Preliminaries

The ring of $m \times m$ matrices with entries belonging to a ring R is denoted by $\mathrm{Mat}_m(R)$. We use the notation $[M]_{i,*}$, $1 \leqslant i \leqslant m$, for the $1 \times m$-matrix which is the ith row of an $m \times m$-matrix M. The notation M^T is used for the transpose of a matrix (vector) M.

If F is a differential field with derivation ∂ then $\mathrm{Const}\,(F) = \{c \in F \mid \partial c = 0\}$ is the *constant field* of F.

2.1 Differential Universal and Adequate Field Extensions

Let K be a differential field of characteristic 0 with derivation $\partial ={}'$.

Definition 1. *An* adequate differential extension Λ *of* K *is a differential field extension* Λ *of* K *such that any differential system*

$$\partial y = Ay, \tag{1}$$

with $A \in \mathrm{Mat}_m(K)$ *has a solution space of dimension* m *in* Λ^m *over* $\mathrm{Const}\,(\Lambda)$.

If $\mathrm{Const}\,(K)$ is algebraically closed then there exists a unique (up to a differential isomorphism) adequate differential extension Λ such that $\mathrm{Const}\,(\Lambda) = \mathrm{Const}\,(K)$ which is called the *universal differential field extension* of K [18, Sect. 3.2]. For any differential field K of characteristic 0 there exists a differential extension whose constant field is algebraically closed. Indeed, this is the algebraic closure $\bar{K}$ with the derivation obtained by extending the derivation of K in the natural way. In this case, $\mathrm{Const}\,(\bar{K}) = \overline{\mathrm{Const}\,(K)}$ (see [18, Exercises 1.5, 2:(c),(d)]). Existence of the universal differential extension for $\bar{K}$ implies that there exists an adequate differential extension for K, i.e., for an arbitrary differential field of characteristic zero.

In the sequel, we denote by Λ a fixed adequate differential extension of K, and we suppose that the vector solutions of systems in the form (2) lie in Λ^m.

In addition to the first-order systems of the form (1), we also consider the differential systems of arbitrary order $r \geqslant 1$. Each of these systems can be represented, e.g., in the form

$$A_r y^{(r)} + A_{r-1} y^{(r-1)} + \cdots + A_0 y = 0, \tag{2}$$

where the matrices $A_0, A_1, \ldots, A_r$ belong to $\mathrm{Mat}_m(K)$, $m \geqslant 1$, and A_r (the *leading matrix* of the system) is non-zero. The system (2) can be written as $L(y) = 0$ where

$$L = A_r \partial^r + A_{r-1} \partial^{r-1} + \cdots + A_0. \tag{3}$$

The number r is the *order* of L (we write $r = \mathrm{ord}\, L$). The operator (3) can be alternatively represented as a matrix in $\mathrm{Mat}_m(K[\partial])$:

$$\begin{pmatrix} L_{11} & \ldots & L_{1m} \\ \ldots & \ldots & \ldots \\ L_{m1} & \ldots & L_{mm} \end{pmatrix}, \tag{4}$$

$L_{ij} \in K\,[\partial]$, $i, j = 1, \dots, m$, with $\max_{i,j} \operatorname{ord} L_{ij} = r$. We say that the operator $L \in \mathrm{Mat}_m(K\,[\partial])$ (as well as the system $L(y) = 0$) is of *full rank*, if the rows $(L_{i1}, \dots, L_{im})$, $i = 1, \dots, m$, of matrix (4) are linearly independent over $K\,[\partial]$. The matrix A_r is the leading matrix of both the system $L(y) = 0$ and operator L, regardless of representation form.

2.2 Universal Differential Extension of Formal Laurent Series Field

Let K_0 be a subfield of the complex number field $\mathbb{C}$ and K be the field $K_0((x))$ of formal Laurent series with coefficients in K_0, equipped with the derivation $\partial = \frac{d}{dx}$. As it is well known [20, Sect. 110], if K_0 is algebraically closed then the universal differential field extension Λ is the quotient field of the ring generated by expressions of the form

$$e^{P(x)} x^{\gamma} (\psi_0 + \psi_1 \ln x + \dots + \psi_s (\ln x)^s), \tag{5}$$

where in any such expression

- $P(x) \in K_0[x^{-1/q}]$, q is a positive integer,
- $\gamma \in K_0$,
- s is a non-negative integer and

$$\psi_j \in K_0[[x^{1/q}]], \tag{6}$$

  $j = 0, 1, \dots, s$.

In fact, system (1) has m linearly independent solutions $b_1(x), \dots, b_m(x)$ such that

$$b_i(x) = e^{P_i(x)} x^{\gamma_i} \Psi_i(x), \tag{7}$$

where the factor $e^{P_i(x)} x^{\gamma_i}$ is common for all components of b_i, and

$\gamma_i \in K_0$, q_i is a positive integer, $P_i(x) \in K_0[x^{-1/q_i}]$, $\Psi_i(x) \in K_0^m[[x^{1/q_i}]][\ln x]$,

$i = 1, \dots, m$.

Definition 2. *Solutions of the form (7) will be called* (formal) exponential-logarithmic *solutions. If* $q = 1$ *and* $P(x) = 0$ *then the solutions (7) are called* regular.

Remark 1. *If K_0 is not algebraically closed then there exists a simple algebraic extension K_1 of K_0 (specific for each system) such that system (1) has m linearly independent solutions of the form (7) with $\gamma_i \in K_1$, $P_i(x) \in K_1[x^{-1/q_i}]$, $\Psi_i(x) \in K_1^m[[x^{1/q_i}]][\ln x]$, $i = 1, \dots, m$.*

2.3 Row Frontal Matrix and Row Order

Let a full-rank operator $L \in \mathrm{Mat}_m(K[\partial])$ be of the form (3). If $1 \leqslant i \leqslant m$ then define $\alpha_i(L)$ as the biggest integer k, $0 \leqslant k \leqslant r$, such that $[A_k]_{i,*}$ is a nonzero row. The matrix $M \in \mathrm{Mat}_m(K)$ such that $[M]_{i,*} = [A_{\alpha_i(L)}]_{i,*}$, $i = 1, \ldots, m$, is the *row frontal matrix* of L. The vector $(\alpha_1(L), \ldots, \alpha_m(L))$ is the *row order* of L. We will write simply $(\alpha_1, \ldots, \alpha_m)$, when it is clear which operator is considered.

Definition 3. *An operator $U \in \mathrm{Mat}_m(K[\partial])$ is* unimodular *(or* invertible*) if there exists $\bar{U} \in \mathrm{Mat}_m(K[\partial])$ such that $\bar{U}U = U\bar{U} = I_m$. An operator in $\mathrm{Mat}_m(K[\partial])$ is* row reduced *if its row frontal matrix is invertible.*

The following proposition is a consequence of [9, Thm. 2.2]:

Proposition 1. *Let $L \in \mathrm{Mat}_m(K[\partial])$ then there exist $U, \check{L} \in \mathrm{Mat}_m(K[\partial])$ such that U is unimodular and $\check{L}$ defined by*

$$\check{L} = UL \tag{8}$$

and represented in the form (4), has k zero rows, where $0 \leqslant k \leqslant m$, and the row frontal matrix of $\check{L}$ is of rank $m - k$ over K. The operator L is of full rank if and only if $k = 0$, and in this case the operator $\check{L}$ in (8) is row reduced.

We will say that the system (2) is unimodular whenever the corresponding matrix (4) is.

3 When K Is a Constructive Field

Definition 4. *A ring (field) K is said to be* constructive *if there exist algorithms for performing the ring (field) operations and an algorithm for zero testing in K*

This definition is close to the definition of an *explicit* field given in [11].

Suppose that K is a constructive field. Then the proof of the already mentioned theorem [9, Thm. 2.2] gives an algorithm for constructing $U, \check{L}$. We will refer to this algorithm as RR (*Row-Reduction*).

3.1 The Dimension of the Solution Space of a Given Full Rank System

Proposition 2. ([1]) *Let $L \in \mathrm{Mat}_m(K[\partial])$ be row reduced, and denote by $\alpha = (\alpha_1, \ldots, \alpha_m)$ its row order. Then the dimension of its solution space V_L is given by: $\dim V_L = \sum_{i=1}^m \alpha_i$.*

Hence, when the field K is constructive we can apply algorithm RR, and compute, by Proposition 2, the dimension of the solution space of a given full-rank system.

Note that in the case when K is the field of rational functions of x over a field of characteristic zero with $\partial = \frac{d}{dx}$, some inequalities close to the formula given in Proposition 2 can be derived from the results of [12].

3.2 Recognizing the Unimodularity of an Operator and Computing the Inverse Operator

The following property of unimodular operators is a direct result of Proposition 2.

Proposition 3. *[2] Let $L \in \mathrm{Mat}_m(K[\partial])$ be of full rank. Then L is unimodular if and only if $\dim V_L = 0$. Moreover, in the case when the row frontal matrix of L is invertible, L is unimodular if and only if $\mathrm{ord}\, L = 0$.*

Algorithm RR allows one to compute a unimodular $U \in \mathrm{Mat}_m(K[\partial])$ such that the operator $\breve{L} = UL$ has an invertible row frontal matrix. Proposition 3 implies that L is unimodular if and only if $\breve{L}$ is an invertible matrix in $\mathrm{Mat}_m(K)$. In this case $(\breve{L})^{-1}UL = I_m$, i.e., $(\breve{L})^{-1}U$ is the inverse of L. Hence the following proposition holds (taking into account Proposition 1, we need not assume that L is of full rank):

Proposition 4. *Let K be constructive and $L \in \mathrm{Mat}_m(K[\partial])$. One can recognize algorithmically whether L is unimodular or not, and compute the inverse operator if it is.*

4 When the Zero Testing Problem in K Is Undecidable

It is easy to see that if the zero testing problem in K is undecidable then the problem of recognizing whether a given $L \in \mathrm{Mat}_m(K[\partial])$ is of full rank is undecidable. Indeed, let $u \in K$, then the operator

$$L = \begin{pmatrix} u\partial & \partial \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} u & 1 \\ 0 & 0 \end{pmatrix} \partial + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

is of full rank if and only if $u \neq 0$, and any algorithm to recognize whether a given $L \in \mathrm{Mat}_m(K[\partial])$ is of full rank can be used for zero testing in K.

Furthermore, it turns out that if the zero testing problem in K is undecidable then even with a prior knowledge that operators under consideration are of full rank, many questions about those operators remain undecidable.

Proposition 5. *Let the zero testing problem in K be undecidable. Then for $m \geqslant 2$ the following problems about a full-rank operator $L \in \mathrm{Mat}_m(K[\partial])$ are undecidable:*
(a) computing $\dim V_L$,
(b) testing unimodularity of L.

Proof. (a) Let $u \in K$ and

$$L = \begin{pmatrix} u\partial + 1 & \partial \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} u & 1 \\ 0 & 0 \end{pmatrix} \partial + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{9}$$

If $u = 0$ then L is unimodular:

$$\begin{pmatrix} 1 & \partial \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -\partial \\ 0 & 1 \end{pmatrix}$$

and, therefore, $\dim V_L = 0$. If $u \neq 0$ then $\dim V_L = 1$ by Proposition 2. We have

$$\dim V_L = \begin{cases} 0 & \text{if } u = 0, \\ 1 & \text{if } u \neq 0. \end{cases}$$

This implies that if we have an algorithm for computing the dimension then we have an algorithm for the zero testing problem.

(b) As we have seen the operator L of the form (9) is unimodular if and only if $u = 0$.

As a consequence of Propositions 4, 5 we have the following:

Testing unimodularity and determining the dimension of the solution space of an arbitrary full-rank system can be done algorithmically if and only if the zero testing problem in K can be solved algorithmically.

One of the general causes of difficulties in the zero testing problem in K may be associated with non-uniqueness of representation of the elements of K [11, Sect. 2]. This is illustrated in Section 5.1.

5 Computable Power Series

5.1 Semi-constructive Fields

Let K be the field $K_0((x))$ where K_0 is a constructive field of characteristic 0. The field K contains the set $K|_{\mathrm{C}}$ of *computable* series, whose sequences of coefficients can be represented algorithmically. That is to say that for each series $a(x) \in K|_{\mathrm{C}}$ there exists an algorithm Ξ_a to compute the coefficient $a_i \in K_0$ for a given i; arbitrary algorithms which are applicable to integer numbers and return elements of K_0 are allowed. For this set to be considered as a constructive differential subfield of K, it would be necessary to define algorithmically on $K|_{\mathrm{C}}$ the field operations of the field K, the unary operation $\frac{d}{dx}$, and a zero testing algorithm as well. However, in accordance with the classical results of Turing [22], we are not able to solve algorithmically the zero testing problem in $K|_{\mathrm{C}}$. As mentioned in Section 4, the undecidability of the zero testing problem is quite often associated with the fact that the elements of the field (or ring) under consideration can be represented in various ways, and for some of which the test is evident while for the others is not. This holds for $K|_{\mathrm{C}}$ as well.

Remark 2. *The field $K|_{C}$ is smaller than the field K because not every sequence of coefficients can be represented algorithmically. Indeed, the set of elements of $K|_{C}$ is countable (each of the algorithms is a finite word in some fixed alphabet) while the cardinality of the set of elements of K is uncountable.*

If the only information we possess about the elements of $K|_{\mathrm{C}}$ is an algorithm to compute their coefficients then the problem of finding the valuation of a given $a(x) \in K|_{\mathrm{C}}$, $\operatorname{val} a(x)$, is undecidable even in the case when it is known in advance that $a(x)$ is not the zero series. This implies that when we work with elements of $K|_{\mathrm{C}}$, i.e., with computable Laurent series, we cannot compute $a^{-1}(x)$ for a given non-zero $a(x) \in K|_{\mathrm{C}}$, since the coefficient of x^{-1} of the series $a'(x)a^{-1}(x) \in K|_{\mathrm{C}}$ is equal to $\operatorname{val} a(x)$, i.e., is equal to the value that we are not able to find algorithmically knowing only Ξ_a. This means that a suitable representation has to contain some additional information besides a corresponding algorithm. The value $\operatorname{val} a(x)$ cannot close the gap, since we have no algorithm to compute the valuation of the sum of two series. However, we can use a lower bound of the valuation instead: observe that if we know that a series $a(x)$ is non-zero then using a valuation lower bound we can compute the exact value of $\operatorname{val} a(x)$. Thus, we can use as the representation of $a(x) \in K|_{\mathrm{C}}$ a pair of the form

$$(\Xi_a, \mu_a), \tag{10}$$

where Ξ_a is an algorithm for computing the coefficient $a_i \in K_0$ for a given i, and the integer μ_a is a lower bound for the valuation of $a(x)$. A computable Laurent series $a(x)$, represented by a pair of the form (10) is equal to $\sum_{i=\mu_a}^{\infty} \Xi_a(i)x^i$.

Of course, there exist other ways to represent computable Laurent series. For example, one can use a pair $(\Xi_a, p_a(x))$, where the algorithm Ξ_a represents a power series that is the regular part of $a(x)$ while $p_a(x) \in K_0[x^{-1}]$ represents explicitly its singular part. We can also represent each Laurent series as a fraction of two power series (the latter are represented algorithmically, this is possible as the field of Laurent series is the quotient field of the ring of power series). So a Laurent series can be represented as a couple $(a(x), b(x))$ of power series with $b(x)$ nonzero.

We can define the field structure on $K|_{\mathrm{C}}$: all field operations can be performed algorithmically. Since we do not have an algorithm for solving the zero testing problem in $K|_{\mathrm{C}}$, we use for $K|_{\mathrm{C}}$ the term "semi-constructive field" instead.

Definition 5. *A ring (field) is* semi-constructive *if there are algorithms to perform the ring (field) operations, but there exists no algorithm to solve the zero testing problem.*

Observe that if the standard representation form is used for rational functions, i.e., for elements in $K_0(x)$, then the field $K_0(x)$ is constructive.

Remark 3. *Consider for the ring $R = K_0[[x]]$ its semi-constructive sub-ring $R|_C$ of computable power series. In this case we do not need to include a lower bound for the valuation into a representation of a series $a(x) \in R|_C$, since 0 is such a bound.*

5.2 Systems with Computable Power Series Coefficients

Below we suppose that K_0 is a constructive field of characteristic 0, $K = K_0((x))$, $R = K_0[[x]]$, and

$$K|_{\mathrm{C}}, \ \ R|_{\mathrm{C}}$$

are a semi-constructive field and, resp., a semi-constructive ring as in Section 5.1. We will consider systems of the form

$$L(y) = 0, \quad L \in \mathrm{Mat}_m\left(R|_{\mathrm{C}}\left[\frac{d}{dx}\right]\right). \tag{11}$$

It follows from Proposition 5 that the problems (a) and (b) listed in that proposition are undecidable if L is as in (11). At first glance, it seems that such undecidability is mostly caused by the inability to distinguish zero and nonzero coefficients of operators and systems. However, even if we know in advance which of the coefficients of an operator L are null, we, nevertheless, cannot solve problems (a) and (b) of Proposition 5 algorithmically. Let $u(x) \in R|_{\mathrm{C}}$ and

$$L = \begin{pmatrix} (u(x)x+1)\frac{d}{dx}+1 & \frac{d}{dx} \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} u(x)x+1 & 1 \\ 0 & 0 \end{pmatrix} \frac{d}{dx} + \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

For such an operator, we know in advance which of its coefficients are equal to zero, but we do not know whether the power series $u(x)$ is equal to zero. It is easy to see that

$$\dim V_L = \begin{cases} 0 & \text{if } u(x) = 0, \\ 1 & \text{if } u(x) \neq 0. \end{cases}$$

5.3 On Formal Exponential-Logarithmic Solutions

In [3,6], it was proven that the problems of existence of Laurent series solutions and regular solutions (see Definition 2) for a given system (11) are decidable. A regular solution has the form $x^\gamma w(x)$, where $\gamma \in \bar{K}_0$, and $w(x) \in \bar{K}_0((x))^m[\ln x]$; in the context of [6], $w(x) \in \left(\bar{K}_0((x))|_{\mathrm{C}}\right)^m[\ln x]$. In those papers, it was proven also that if non-zero Laurent series or regular solutions exist then we can construct them, i.e., find a lower bound for valuations of all involved Laurent series as well as any number of terms of the series; for regular solutions we also find the corresponding values of γ, the degrees of polynomials in $\ln x$ etc. It was shown also that instead of $\bar{K}_0$ which is the algebraic closure of K_0 some simple algebraic extension K_1 of K_0 may be used.

Remark 4. *The power series which appear in [3,6] as coefficients of a given system can be represented not only by algorithms as described above but also as "black boxes", i.e., by procedures of unknown internal form.*

Proposition 6. *Let m be an integer, $m \geqslant 2$, and K_0 be a constructive subfield of $\mathbb{C}$. Then for a given full-rank system of the form (11),*

(i) the question whether nonzero Laurent series solutions exist as well as the question whether nonzero regular solutions exist are algorithmically decidable;

(ii) the question whether nonzero formal exponential-logarithmic solutions exist is algorithmically undecidable;

(iii) the question whether nonzero formal exponential-logarithmic solutions which are not regular solutions exist is algorithmically undecidable.

Proof. (i) This follows from [3,6], as it was explained in the beginning of this section.

(ii) A given L is unimodular if and only if the system (11) has no non-zero formal exponential-logarithmic solution, and the claim follows from Proposition 5 (problem (b)).

(iii) A full-rank operator L is evidently unimodular if and only if it has no regular solution and no exponential-logarithmic solution which is not regular. By (i), we can test whether the system $L(y) = 0$ has no regular solution. Thus, if we are able to test whether this system has no exponential-logarithmic solution which is not regular then we can test whether L is unimodular or not. However, this is an undecidable problem by Proposition 5 (problem (b)).

Proposition 7. *Let m be an integer number, $m \geqslant 2$, K_0 be a constructive subset of $\mathbb{C}$. Let $L(y) = 0$ be a full-rank system of the form (11), and $d = \dim V_L$. Then V_L has a basis $b_1(x), \ldots, b_d(x)$ consisting of exponential-logarithmic solutions such that any $\Psi_i(x)$ from (7) is of the form $\Psi_i(x) = \Phi_i(x^{1/q_i})$ where q_i is a non-negative integer,*

$$\Phi_i(x) \in ((K_1[[x]])\,|_c)^m\,[\ln x], \tag{12}$$

and K_1 is a simple algebraic extension of K_0. In addition to (12), $\gamma_i \in K_1$, $P_i(x) \in K_1[x^{-1/q_i}]$, $i = 1, \ldots, d$.

Proof. It follows from, e.g., [4,5,8], that for any operator L of full rank there exists an operator F such that the leading matrix of FL is invertible. The system $FL(y) = 0$ is equivalent to a first order system of the form $y' = Ay$, $A \in \mathrm{Mat}_{ms}(K((x)))$, $s = \mathrm{ord}\, FL$. It is known ([7]) that for a first-order system there exists a simple algebraic extension K_1 of K_0 such that those γ_i and the coefficients of $P_i(x)$ which appear in its solutions of the form (7), belong to K_1. The field K_1 is constructive since K_0 is. Obviously, $q_i \in \mathbb{N}$.

The substitution

$$x = t^{q_i}, \;\; y(t^{q_i}) = z(t)e^{P_i(t^{q_i})},$$

$P_i(t^{q_i}) \in K_1[1/t]$, into the original system $L(y) = 0$ transforms it into a full-rank system which can be represented as

$$\tilde{L}(z) = 0, \;\; \tilde{L} \in \mathrm{Mat}_m\left((K_1[[t]])|_{\mathrm{c}}\left[\frac{d}{dt}\right]\right).$$

The Laurent series that appear in the regular solutions of this new system can be taken to be computable, as it follows from [3,6] (see the beginning of this section).

Thus, the series that appear in the representation of solutions are computable (Proposition 7), but we cannot find them algorithmically (Proposition 6). In fact, Proposition 7 guarantees existence. However, the operator F mentioned therein cannot be constructed algorithmically.

Remark 5. *In the case of first-order systems of the form (1), the questions formulated in Proposition 6(ii, iii) are decidable. This follows from the fact that* *for constructing exponential-logarithmic solutions of a system of this form one needs only a finite number of terms of the entries (which are Laurent series) of A, and the number of those terms can be computed in advance ([15,7,17]). This holds also for higher-order systems whose leading matrices are invertible.*

It is proven ([19]) that if the dimension d of the space of exponential-logarithmic solutions is known in advance then the basis $b_1, \dots, b_d$ which is mentioned in Proposition 7 can be constructed algorithmically. The corresponding algorithm is implemented in Maple.

As we see, if the algorithmic representation of series is used and if arbitrary algorithms representing series are admitted then some of the problems related to linear ordinary differential systems are decidable, while others are not. There is a subtle border between them, and a careful formulation of each of the problems under consideration is absolutely necessary. A small change in the formulation of a decidable problem can transform it into an undecidable one, and vice versa.

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