On the Dimension of Solution Spaces of Full Rank Linear Differential Systems

S.A. Abramov[1,⋆] and M.A. Barkatou[2]

[1] Computing Centre of the Russian Academy of Sciences, Vavilova, 40, Moscow 119333, Russia
sergeyabramov@mail.ru

[2] Institut XLIM, Département Mathématiques et Informatique, Université de Limoges, CNRS, 123, Av. A. Thomas, 87060 Limoges Cedex, France
moulay.barkatou@unilim.fr

Abstract. Systems of linear ordinary differential equations of arbitrary orders of full rank are considered. We study the change in the dimension of the solution space that occurs while differentiating one of the equations. Basing on this, we show a way to compute the dimension of the solution space of a given full rank system. In addition, we show how the change in the dimension can be used to estimate the number of steps of some algorithms to convert a given full rank system into an appropriate form.

1 Introduction

Given a system of linear homogeneous differential equations with the coefficients from some "functional" field. Suppose we differentiate one of its equations. What would then happen to the solution space of the system? Would it remain unchanged or would we always get some extra solutions?

In the scalar case, when we differentiate equation $L(y) = 0$, the resulting equation $(L(y))' = 0$ has a larger order than the original one. Let the coefficients of the equations belong to some differential field $\mathbb{K}$, and the solutions be in some "functional" space Λ. If $\mathbb{K}$ and Λ are such that every equation of order m has a solution space of dimension m then equation $(L(y))' = 0$ has more solutions than equation $L(y) = 0$.

For the systems of linear ordinary differential equations the problem is not as simple as for scalar equations. Indeed, the solution space of a system of equations is the intersection of the solution spaces of all the equations of the system. Thus, the fact the solution spaces of the individual equations becomes larger does not imply that their intersection becomes larger too.

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In the present paper we prove that differentiating one of the equations in a full rank system, one increases the dimension of the solution space by one (Section 3). In Section 4 some applications of this are discussed.

In Appendix the difference case is briefly considered.

2 Preliminaries

The ring of $m \times m$ matrices with entries in a ring R is denoted by $\mathrm{Mat}_m(R)$. By I_m we denote the identity matrix of order m. The notation M^T is used for the transpose of a matrix (vector) M.

Let $(\mathbb{K}, \partial)$, $\partial ='$, be a differential field of characteristic 0 with an algebraically closed constant field $Const(\mathbb{K}) = \{c \in \mathbb{K} \mid \partial c = 0\}$. We denote by Λ a fixed *universal differential extension field* of $\mathbb{K}$ (see [9, Sect. 3.2]). This is a differential extension Λ of $\mathbb{K}$ with $Const(\Lambda) = Const(\mathbb{K})$ such that any differential system

$$\partial y = Ay, \tag{1}$$

with $A \in \mathrm{Mat}_m(\mathbb{K})$ has a solution space of dimension m over the constants.

If, e.g., $\mathbb{K}$ is a subfield of the field $\mathbb{C}((x))$ of formal Laurent series with complex coefficients with $\partial = \frac{d}{dx}$ then we can consider Λ as the quotient field of the ring generated by expressions of form $e^{P(x)}x^{\gamma}(\psi_0 + \psi_1 \log x + \cdots + \psi_s (\log x)^s)$, where in any such expression

- $P(x)$ is a polynomial in $x^{-1/p}$, where p is a positive integer,
- $\gamma \in \mathbb{C}$,
- s is a non-negative integer and $\psi_i \in \mathbb{C}[[x^{1/p}]]$, $i = 0, 1, \ldots, s$.

Besides first-order systems of form (1) we will consider differential systems of order $r \geqslant 1$ which have the form

$$A_r y^{(r)} + A_{r-1} y^{(r-1)} + \cdots + A_0 y = 0. \tag{2}$$

The coefficient matrices

$$A_0, A_1, \ldots, A_r \tag{3}$$

belong to $\mathrm{Mat}_m(\mathbb{K})$, and A_r (the *leading matrix* of the system) is non-zero.

Remark 1. *If A_r is invertible in $\mathrm{Mat}_m(\mathbb{K})$ then the system (2) is equivalent to the first order system having mr equations: $Y' = AY$, with*

$$A = \begin{pmatrix} 0 & I_m & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & I_m \\ \hat{A}_0 & \hat{A}_1 & \ldots & \hat{A}_{r-1} \end{pmatrix}, \tag{4}$$

where $\hat{A}_k = -A_r^{-1} A_k$, $k = 0, 1, \ldots, r-1$, and

$$Y = \left(y_1 \ldots, y_m,\, y_1' \ldots, y_m', \ldots,\, y_1^{(r-1)}, \ldots, y_m^{(r-1)}\right)^T. \tag{5}$$

Therefore if the leading matrix of the system (2) is invertible then the dimension of the solution space of this system is equal to mr.

Denote the ring $\mathrm{Mat}_m(\mathbb{K}[\partial])$ by $\mathcal{D}_m$. System (2) can be written as $L(y) = 0$ where

$$L = A_r\partial^r + A_{r-1}\partial^{r-1} + \cdots + A_0 \in \mathcal{D}_m. \tag{6}$$

System (2) can be also written as a system of m scalar linear equations

$$L_1(y_1, \ldots, y_m) = 0, \ \ldots, \ L_m(y_1, \ldots, y_m) = 0, \tag{7}$$

with

$$L_i(y_1, \ldots, y_m) = \sum_{j=1}^{m} l_{ij}(y_j), \ \ l_{ij} \in \mathbb{K}[\partial], \ \ i, j = 1, \ldots, m, \quad \max_{i,j} \operatorname{ord} l_{ij} = r. \tag{8}$$

When a system is represented in form (7) we can rewrite it in form (2) and vice versa. The matrix A_r is the leading matrix of the system regardless of representation form. We suppose also that the system is of *full rank*, i.e., that equations (7) are independent over $\mathbb{K}[\partial]$, in other words the rows

$$\ell_i = (l_{i1}, \ldots, l_{im}), \tag{9}$$

$i = 1, \ldots, m$, are linearly independent over $\mathbb{K}[\partial]$. We say that an operator $L \in \mathcal{D}_m$ is of full rank if the system $L(y) = 0$ is. The leading matrix of L is the leading matrix of the system $L(y) = 0$.

3 Differentiating of an Equation of a Full Rank System

3.1 Formulation of the Main Theorem

Our nearest purpose is to prove the following theorem:

Theorem 1. *Let a system of the form (7) be of full rank. Let the system*

$$L_1(y_1, \ldots, y_m) = 0, \ \ldots, \ L_{m-1}(y_1, \ldots, y_m) = 0, \ \tilde{L}_m(y_1, \ldots, y_m) = 0, \tag{10}$$

be such that its first $m-1$ equations are as in the system (7) while the m-th equation is the result of differenting of the m-th equation of (7), thus the equation $\tilde{L}_m(y_1, \ldots, y_m) = 0$ is equivalent to the equation $(L_m(y_1, \ldots, y_m))' = 0$. Then the dimension of the solution space of (10) exceeds by 1 the dimension of the solution space of (7).

To prove this theorem we consider first the case when a given system has an invertible leading matrix (in this case the system is certainly of full rank). After this we consider the general case of a system of full rank.

The set of solutions of (10) coincides with the union of the set of solutions of all the systems

$$L_1(y_1, \ldots, y_m) = 0, \ \ldots, \ L_{m-1}(y_1, \ldots, y_m) = 0, \ L_m(y_1, \ldots, y_m) = c, \tag{11}$$

when c runs through the set of constants of Λ (any constant c specifies a system). Note that the fact that the dimension of the solution space of (10) does not exceed the dimension of the solution space of (7) more than by 1 is trivial: if c_1, c_2 are constants and $\varphi, \psi \in \Lambda^m$ are solutions of the system (11) with $c = c_1$, resp. $c = c_2$, then $c_2\varphi - c_1\psi$ is a solution of (7). Thus it is sufficient to prove simply that the differentiation increases the dimension of the solution space of a full rank system.

It is also trivial that if (7) is of full rank then (10) is also of full rank. Going back to (8), (9), let $\tilde{\ell}_m = \partial l_m = (\tilde{l}_{m1}, \dots, \tilde{l}_{mm})$. If $u_1, \dots, u_m \in \mathbb{K}[\partial]$ are such that $u_1\ell_1 + \cdots + u_{m-1}\ell_{m-1} + u_m\tilde{\ell}_m = 0$ then $v_1\ell_1 + \cdots + v_{m-1}\ell_{m-1} + v_m\ell_m = 0$ where $v_1 = u_1, \dots, v_{m-1} = u_{m-1}$, $v_m = u_m\partial$, and if $u_i \neq 0$, $0 \leqslant i \leqslant m$, then $v_i \neq 0$.

3.2 Invertible Leading Matrix Case

Lemma 1. *Let the leading matrix of (7) be invertible. Then the dimension of the solution space of (10) is larger than the dimension of the solution space of (7).*

Proof. Together with the union of the set of solutions of all the systems (11) when c runs through the set of constants of Λ, we consider the system

$$L_1(y_1, \dots, y_m) = 0, \;\; \dots, \;\; L_{m-1}(y_1, \dots, y_m) = 0, \qquad L_m(y_1, \dots, y_m) = y_{m+1}, \;\; y'_{m+1} = 0. \qquad (12)$$

Observe that the system (12) is equivalent to the system $\tilde{Y}' = \tilde{A}\tilde{Y}$ where the matrix $\tilde{A} \in \mathrm{Mat}_{rm+1}(\mathbb{K})$ is obtained from the matrix (4) by adding the last row of zeros and the last column $(0, \dots, 0, 1, 0)^T$. The column vector $\tilde{Y}$ is obtained from Y (see (5)) by adding y_{m+1} as the last component. The dimension of the solution space of $\tilde{Y}' = \tilde{A}\tilde{Y}$ is equal to $mr+1$, while the dimension of the solution space of the original system is equal to mr (Remark 1).

Therefore the system (12) has a solution $(\tilde{y}_1, \dots, \tilde{y}_m, \tilde{y}_{m+1})$ with $\tilde{y}_{m+1} \neq 0$. Evidently $(\tilde{y}_1, \dots, \tilde{y}_m)$ is a solution of (11) with $c = \tilde{y}_{m+1} \neq 0$, but $(\tilde{y}_1, \dots, \tilde{y}_m)$ is not a solution of (7) since $L_m(\tilde{y}_1, \dots, \tilde{y}_m) \neq 0$. The claim follows.

Thus the dimension of the solution space of (10) is equal to $mr + 1$. This proves the Theorem 1 in the case when the given system has an invertible leading matrix.

3.3 General Case of a System of Full Rank

In [2] the following proposition has been proved:

Proposition 1. *Let L be a full rank operator of the form (6). Then there exists $N \in \mathcal{D}_m$ such that the leading matrix of LN is invertible. (In addition, N can be taken such that LN is of order r).*

Using Lemma 1 and Proposition 1 we can complete the proof of Theorem 1. Let a given system of the form (2) be represented as $L(y) = 0$ where L is as in (6). If the leading matrix A_r of L is invertible then the statement of the theorem follows from Lemma 1. Otherwise let N be an operator such that the leading matrix of LN is invertible (Proposition 1). Set

$$D = \begin{pmatrix} 0 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & 1 \end{pmatrix} \partial + \begin{pmatrix} 1 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 1 & 0 \\ 0 & \dots & 0 & 0 \end{pmatrix}, \tag{13}$$

$D \in \mathcal{D}_m$.

By Lemma 1 the dimension of the solution space of the system $DLN(y) = 0$ is larger than the dimension of the solution space of the system $LN(y) = 0$. This implies that there exists $\varphi \in \Lambda^m$ such that $N(\varphi)$ is a solution of the system $DL(y) = 0$ but is not a solution of $L(y) = 0$. In turn this implies that the dimension of the solution space of the system $DL(y) = 0$ is larger than the dimension of the solution space of the system $L(y) = 0$. Theorem 1 is proved.

Remark 2. *Theorem 1 is valid for the case of a full rank inhomogeneous system as well. That is a system of the form* $L(y) = b$*, with* $L \in \mathcal{D}_m$ *of full rank and* $b \in \mathbb{K}^m$*. First of all note that this system has at least one solution in* Λ^m *since by adding to* y *an* $(m+1)$*-st component with value* 1*, one can transform the given system into a homogeneous system with a matrix belonging to* $\mathrm{Mat}_{m+1}(\mathbb{K})$*. The set of solutions in* Λ^m *of* $L(y) = b$ *is an affine space over the* $Const(\Lambda)$ *and is given by* $V_L + f$ *where* $V_L \subset \Lambda^m$ *is the solution space of the homogeneous system* $L(y) = 0$ *and* $f \in \Lambda^m$ *is a particular solution of* $L(y) = b$*. When we differentiate the* m*-th equation of the system* $L(y) = b$ *we get a new system* $\tilde{L}(y) = \tilde{b}$ *where the operator* $\tilde{L}$ *corresponds to system (10). By Theorem 1* $\dim V_{\tilde{L}} = \dim V_L + 1$.

4 Some Applications

4.1 The Dimension of the Solution Space of a Given Full Rank System

By Remark 1, if the leading matrix of the system (2) is invertible then the dimension of the solution space of this system is equal to mr. How to find the dimension of the solution space in the general case?

We use the notation

$$[M]_{i,*}\,, \quad 1 \leqslant i \leqslant m,$$

for the $(1 \times m)$-matrix which is the i-th row of an $(m \times m)$-matrix M. Let a full rank operator $L \in \mathcal{D}_m$ be of the form (6). If $1 \leqslant i \leqslant m$ then define $\alpha_i(L)$ as the maximal integer k, $1 \leqslant k \leqslant r$, such that $[A_k]_{i,*}$ is a nonzero row. The matrix $M \in \mathrm{Mat}_m(\mathbb{K})$ such that $[M]_{i,*} = [A_{\alpha_i(L)}]_{i,*}$, $i = 1, 2, \dots, m$, is the *row frontal matrix* of L.

Theorem 2. *Let the row frontal matrix of a full rank system $L(y) = 0$, $L \in \mathcal{D}_m$, be invertible. Then the dimension of the solution space of this system is $\sum_{i=1}^{m} \alpha_i(L)$.*

Proof. It follows directly from Theorem 1: when we differentiate $r - \alpha_i(L)$ times the i-th equation of the given system, $i = 1, 2, \ldots, m$, we increase the dimension of the solution space by $mr - \sum_{i=1}^{m} \alpha_i(L)$, and the received full rank system has the leading matrix which coincides with the row frontal matrix of the original system, therefore the obtained system has an invertible leading matrix and the dimension of its solution space is equal to mr.

In [6,7] algorithms to convert a given full rank system into an equivalent system having an invertible row frontal matrix were proposed. It is supposed that the field $\mathbb{K}$ is constructive, in particular that there exists a procedure for recognizing whether a given element of $\mathbb{K}$ is equal to 0. Therefore in such situations we are able to compute the dimension of the solution space of a given full rank system.

4.2 Faster Computation of *l*-Embracing Systems

Suppose that $\mathbb{K} = K(x)$, $\partial = \frac{d}{dx}$, where K is a field of characteristic zero such that each of its elements is a constant. For any system S of the form (2) the algorithm EG_δ ([4,5]) constructs an *l-embracing* system $\bar{S}$:

$$\bar{A}_r(x)y^{(r)}(x) + \cdots + \bar{A}_1(x)y'(x) + \bar{A}_0(x)y(x) = 0,$$

of the same form, but with the leading matrix $\bar{A}_r(x)$ being invertible, and with the solution space containing all the solutions of S. EG_δ is used for finding a finite super-set of the set of singular points of solutions of the given system.

First we describe briefly the algorithm EG_δ, and then discuss its improvement which is due to Theorem 1.

Let the i-th row of the matrix $A_s(x)$, $0 \leqslant s \leqslant r$, be nonzero and the i-th rows of the matrices $A_{s-1}(x), A_{s-2}(x), \ldots, A_0(x)$ be zero. Let the t-th entry, $1 \leqslant t \leqslant m$, be the last nonzero entry of the i-th row of $A_s(x)$. Then, the number $(r - s) \cdot m + t$ is called the *length* of the i-th equation of the system, and the entry of matrix $A_s(x)$ having indices i, t is called the *last nonzero coefficient* of the i-th equation of the system.

Algorithm EG_δ is based on alternation of reductions and differential shifts. Let us explain how the reduction works. It is checked whether the rows of the leading matrix are linearly dependent over $K(x)$. If they are, coefficients of the dependence $v_1(x), v_2(x), \ldots, v_m(x) \in K[x]$ are found. From the equations of the system corresponding to nonzero coefficients, we select the equation of the greatest length. Let it be the i-th equation. This equation is replaced by the linear combination of the equations with the coefficients $v_1(x), v_2(x), \ldots, v_m(x)$. As a result, the i-th row of the leading matrix vanishes. This step is called *reduction* (the reduction does not increase lengths of the equations).

Let the i-th row of the leading matrix be zero, and let $a(x)$ be the last nonzero coefficient of the i-th equation. Let us divide this equation by $a(x)$, differentiate it, and clear the denominators. This operation is called *differential shift* of the i-th equation of the system. Due to the performed division by the last nonzero coefficient, this operation decreases the length of the i-th equation in the system (2).

The algorithm EG_δ is as follows. If the rows of the leading matrix are linearly dependent over $K(x)$, then the reduction is performed. Suppose that this makes the i-th row of the leading matrix zero. Then, we perform the differential shift of the i-th equation and continue the process of alternated reductions and differential shifts until the leading matrix becomes nonsingular. (We never get the equation $0 = 0$ since the equations of the original system are independent over $K(x)\,[\partial]$.)

As we have mentioned no single equation increases its length due to the reduction. The differential shift decreases the length of the corresponding equation. Thus the sum of all the lengths is decreased by a "reduction + differential shift" step. This implies that algorithm EG_δ always terminates and the number of "reduction + differential shift" steps does not exceed $(r+1)m^2$.

Note that the division by the last coefficient of an equation before differentiating the equation is produced to ensure decreasing of the length of the equation. This division and clearing the denominators after the differentiation are quite expensive. If we exclude this division then the cost of a step "reduction + differentiation" will be in general significantly less than the cost of a "reduction + differential shift" step. By Theorem 1 the corresponding sequence of "reduction + differentiation" steps will be finite (thus the new version of EG_δ terminates for any system of the form (2)) and the number k of the "reduction + differentiation" steps does not exceed mr. (By Theorem 1 the dimension of the solution space of the original system is equal to $mr - k$; thus we have one more way to compute the dimension of the solution space besides the one given by Theorem 2). Note that Theorem 1 is applicable since $K(x) \subset \bar{K}(x)$ where $\bar{K}$ is the algebraic closure of K.

This improvement trick works also in the case of an inhomogeneous system when the corresponding homogeneous system is of full rank. The corresponding homogeneous system is transformed independently on the right-hand side when we produce "reduction + differentiation" steps. Therefore the dimension of the solution space of the corresponding homogeneous system increases due to every "reduction + differentiation" step. The upper bound mr keeps valid in the inhomogeneous case (one can also use Remark 2 for proving this).

In addition, we note that due to Theorem 1 it is not necessary to select an equation of maximal length in the reduction substep; therefore various strategies of a row selection on the reduction substep of each "reduction + differentiation" step are possible. Such strategies make it possible to slow down the growth of degrees of system coefficients when applying EG_δ (due to Appendix this works also in the case of difference systems, i.e. gives an improvement of EG_σ [1,3,5]).

Appendix: The Difference Case

A statement similar to Theorem 1 is valid in the difference case, when $(\mathbb{K}, \sigma)$ is a difference field (σ is an automorphism of $\mathbb{K}$) of characteristic 0 with an algebraically closed constant field $Const(\mathbb{K}) = \{c \in \mathbb{K} \mid \sigma c = c\}$. Let Λ the universal Picard-Vessiot ring extension of $\mathbb{K}$ (see [8, Sect. 1.4]). A system is of full rank if its equations are independent over the ring $\mathbb{K}[\sigma]$. The application of the operator $\Delta = \sigma - 1$ is used instead of the differentiation of an equation of a given full rank system.

The proof is a little more complicated since the invertibility of A is needed to guarantee that the dimension space of a system $\sigma y = Ay$, $A \in \mathrm{Mat}_m(\mathbb{K})$, is equal to m . However there is no problem with proving the analog of Lemma 1. After applying Δ to the last equation of the original system we get the system $\sigma\tilde{Y} = \tilde{A}\tilde{Y}$ where the matrix $\tilde{A} \in \mathrm{Mat}_{rm+1}(\mathbb{K})$ is obtained from the matrix (4) by adding the last row $(0, \ldots, 0, 1)$ and the last column $(0, \ldots, 0, 1, 1)^T$. If A is invertible then $\tilde{A}$ is invertible too. Similarly to the differential case the column vector $\tilde{Y}$ is obtained from Y (see (5)) by adding y_{m+1} as the last component.

Consider a system of order $r \geqslant 1$ which has the form

$$A_r\sigma^r y + A_{r-1}\sigma^{r-1}y + \cdots + A_0 y = 0. \tag{14}$$

The coefficient matrices $A_0, A_1, \ldots, A_r$ belong to $\mathrm{Mat}_m(\mathbb{K})$, and if A_r, A_0 (the leading and *trailing* matrices of the system) are invertible then the system (14) is equivalent to the first order system having mr equations: $\sigma Y = AY$, with A as in (4), and A is invertible since $\det A = -\det \hat{A}_0 = \det A_r^{-1} \det A_0 \neq 0$. Therefore if both the leading and trailing matrices of the system (14) are invertible then the dimension of the solution space of this system is equal to mr.

Denote the ring $\mathrm{Mat}_m(\mathbb{K}\,[\sigma])$ by $\mathcal{E}_m$. System (14) can be written as $L(y) = 0$ where

$$L = A_r\sigma^r + A_{r-1}\sigma^{r-1} + \cdots + A_0 \in \mathcal{E}_m. \tag{15}$$

Similarly to the differential case, we say that the operator $L \in \mathcal{E}_m$ is of full rank if the system $L(y) = 0$ is of full rank.

It can be shown (see [5, Sect. 3.5]) that for any full rank operator L of the form (15) there exists $F \in \mathcal{E}_m$ such that the product FL is an operator of order $r+1$ with both the leading and trailing matrices are invertible. Using adjoint difference operators we can analogously to the differential case prove that there exists $N \in \mathcal{E}_m$ such that the operator LN has invertible both the leading and trailing matrices (N can be taken such that LN is of order $r+1$). We can consider the operator D which is obtained from (13) by replacing ∂ by $\Delta = \sigma - 1$ and repeat the reasoning given in the last paragraph above Remark 2.

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