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A direct algorithm to compute rational solutions of first order linear q-difference systems

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Abstract

We present an algorithm to compute rational function solutions to a first order system of linear q-difference equations with rational coefficients. We make use of the fact that q-difference equations bear similarity to differential equations at the point 0 and to difference equations at other points. This allows the combining of known algorithms for the differential and the difference cases. This algorithm does not require preliminary uncoupling of the given system.

Résumé

Nous présentons un algorithme calculant les fonctions rationnelles solutions d'un système du premier ordre d'équations aux q-différences linéaires à coefficients rationnels. Nous utilisons le fait que les équations aux q-différences ont des similarités avec les équations différentielles au point 0 et avec les équations aux différences aux autres points. Cela permet de combiner les algorithmes connus pour ces deux types d'équations. Cet algorithme ne requiert pas un découplage préalable du système donné.

Keywords: Direct algorithms; Linear q-difference equations and systems; Polynomial and rational solutions; Universal denominator

1. Introduction

Let K be a computable field of characteristic zero, $q \in K$ a nonzero element which is not a root of unity, and x transcendental over K.

A system of first order linear q-difference equations with rational coefficients over the field K is a system of the form

$$\begin{aligned} p_1(x)y_1(qx) &= a_{11}(x)y_1(x) + \cdots + a_{1m}(x)y_m(x) + b_1(x) \\ &\vdots \\ p_m(x)y_m(qx) &= a_{m1}(x)y_1(x) + \cdots + a_{mm}(x)y_m(x) + b_m(x). \end{aligned} \tag{1}$$

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We assume the coefficients $p_i(x), a_{ij}(x), b_j(x)$ to be polynomials over the field K.

q-Calculus and the theory and algorithms for linear q-difference equations are of interest in combinatorics, especially in the theory of partitions [10,11]. In this paper, we solve the problem of computing all rational solutions $y(x)=(y_1(x),\ldots,y_m(x))\in K(x)^m$ of (1). Algorithms for solving this problem in the scalar case (that is the case of a single scalar linear q-difference equation of arbitrary order) have been proposed in [6,1]. The algorithmic study for the system case is, in general, less well-developed.

The traditional computer algebra approach of solving linear functional systems is via the cyclic vector method, or other similar elimination methods [14,9]. They first convert the given systems to scalar equations (such a procedure is called the *uncoupling*). Gröbner bases technique can also be used to reduce a recurrent system to the uncoupled form [15]. The major, and well-known problem of this approach is the increase in size of the coefficients of the equations. This makes these approaches applicable only to systems of very small dimension. In this paper, we present an alternative approach (a direct method) to solve the problem for the q-difference case.

It should be noted that there is some progress in solving the analogous problem in the differential and the difference cases: direct methods have been proposed in [13] and in [4,17]. The methods [4,17] are also applicable to the q-difference case, except for the situation where the denominators of some $y_i(x)$ are divisible by x.

We will show below that by combining both differential and difference approaches, it is possible to solve the problem in the q-difference case completely. A characteristic feature of q-difference equations is that they are similar to differential equations near the point 0 and to difference equations near other points. This fact was used in the scalar case in [1].

Similar to the differential and difference cases, the construction proceeds in two steps. In the first step, we construct a *universal denominator*, i.e., a polynomial $U(x)\in K[x]$ such that for all $y(x)\in K(x)^m$, if $y(x)$ is a solution of (1), then $U(x)y(x)$ is a polynomial vector. In the second step, the substitution

$$y(x)=\frac{1}{U(x)}z(x) \tag{2}$$

into (1) reduces the problem to finding polynomial solutions of a system in $z(x)$ of the same type as (1).

If the matrix

$$\begin{pmatrix} a_{11} & \ldots & a_{1m} \\ \vdots & & \vdots \\ a_{m1} & \ldots & a_{mm} \end{pmatrix}, \tag{3}$$

which corresponds to (1) is not invertible over $K(x)$, then there exists a non-trivial linear combination over $K[x]$ of the rows of this matrix that is equal to zero. Suppose that the mth coefficient of this combination is a non-zero polynomial. Then, the last equation of the original system can be replaced by an equation of the form

$$f_1(x)y_1(qx) + \cdots + f_m(x)y_m(qx) = f_{m+1}(x),$$

$f_1(x), \ldots, f_{m+1}(x) \in K(x)$. After the substitution $q^{-1}x$ for x we get

$$h_1(x)y_1(x) + \cdots + h_m(x)y_m(x) = h_{m+1}(x),$$

$h_1(x), \ldots, h_{m+1}(x) \in K(x)$. The coefficient $h_m(x)$ is not equal to zero, and we can eliminate $y_m(x)$ in the first $m-1$ equations of the original system. This gives us a system with a reduced number of unknowns. If the matrix of this system is not invertible over $K(x)$, then we apply the described procedure again. Hence, for the rest of this paper, we can assume that the matrix (3) is invertible.

From time to time we will need to find the largest non-negative integer n such that q^n is a root of a given polynomial with coefficients in K. Therefore, we assume that K is a *q-suitable field*, meaning that there exists an algorithm which, for a given $p \in K[x]$, finds all non-negative integer n such that $p(q^n) = 0$. For instance, if $K = k(q)$ where q is transcendental over k, we can proceed as follows: Let $p(x) = \sum_{i=0}^{d} c_i x^i$ where $c_i \in k[q]$. Compute $s = \min\{i;\ c_i \neq 0\}$ and $t = \max\{j;\ q^j \mid c_s\}$. Then, $p(q^n) = 0$ only if $n \leqslant t$, and the set of all such n can be found by testing the values $n = t, t-1, \ldots, 0$ [7]. An algorithm for solving the equation $p(q^n) = 0$ in the case where q is an algebraic number is considered in [5].

A preliminary version of this paper has appeared as [3].

2. Universal denominators

2.1. Factors other than x

Here, we show the q-modification of algorithm [4]. Algorithm [17] also allows such a modification. In this paper, we do not compare algorithms [4,17], and choose the former simply because its description is shorter.

We name the modification qUD. First of all we set $A(x)$ to be equal to

$$\mathrm{lcm}(p_1(q^{-1}x), \ldots, p_m(q^{-1}x))$$

and compute $B(x)$ as lcm of the denominators of the elements of the matrix inverse of (3). Then compute $q\mathrm{UD}(A, B)$. Let the polynomial $V(x)$ be the result of the computation:

$V(x) := 1;$
$R(n) := Res_x(A(x), B(q^n x));$
`if` $R(n)$ has some non-negative integer roots `then`

$N :=$ the largest non-negative integer root of $R(n)$;
`for` $i = N, N-1, \ldots, 0$ `do`
  $d(x) := gcd(A(x), B(q^i x))$;
  $A(x) := A(x)/d(x)$;
  $B(x) := B(x)/d(q^{-i}x)$;
  $V(x) := V(x)d(x)d(q^{-1}x)\cdots d(q^{-i}x)$
`od`
`fi.`

If the rational functions $F_1(x), \ldots, F_m(x)$ are such that

$$\begin{array}{l} p_1(x)F_1(qx) - a_{11}(x)F_1(x) - \cdots - a_{1m}(x)F_m(x) \\ \vdots \\ p_m(x)F_m(qx) - a_{m1}(x)F_1(x) - \cdots - a_{mm}(x)F_m(x), \end{array} \tag{4}$$

are polynomials and $A(x), B(x)$ are not divisible by x, then the denominators of $F_1(x), \ldots, F_m(x)$ (taken in the reduced form) divide $V(x)$.

The proof is similar to that of Theorem 3 in [4]. All arguments that were given n [4] will still hold if we replace the shifts

$$x \to x + i, \quad x \to x - i,$$

where i is a non-negative integer by the q-shifts

$$x \to q^i x, \quad x \to q^{-i} x$$

and if we ignore the factor x when we consider the irreducible factors of the polynomials $A(x)$ and $B(x)$.

Another approach to work with the denominators is as follows. Let the polynomials $A(x), B(x)$, the non-negative integer N be computed as at the beginning of qUD, and $d(x) = \gcd(A(x), B(q^N x))$. Set $f(x) = d(x)$ if $N = 0$, and $f(x) = d(x)d(q^{-N}x)$ otherwise. After the substitution

$$y(x) = \frac{1}{f(x)} z(x)$$

into (1), we obtain a new system. We can again find $A(x), B(x)$ and write down the equation $R(n) = 0$. It is possible to show that this equation cannot have any integer root $\geqslant N$. Thereby, at some step such a substitution gives us a system such that the corresponding equation $R(n) = 0$ has no non-negative integer root. Such a system cannot have non-polynomial rational solutions. Therefore, it is sufficient to find the polynomial solutions of the obtained system and execute the inverted substitutions.

We would like to emphasize that the algorithm qUB and the one just described, both work only if $A(x)$ and $B(x)$ are not divisible by x. The factor x has to be considered separately since the polynomial x and qx are not relatively prime over K, though any other irreducible polynomial $r(x)$ is relatively prime to $r(qx)$.

2.2. *A bound for the exponent of x*

In the general case, the components of any rational solution $y(x) = (y_1(x), \dots, y_m(x))$ of (1) can be represented in the form

$$y_i(x) = \frac{f_i(x)}{g_i(x)} + \frac{l_{i1}}{x} + \frac{l_{i2}}{x^2} + \cdots + \frac{l_{ih_i}}{x^{h_i}}, \quad g_i(0) \neq 0, \;\; i = 1, \dots, m. \tag{5}$$

The substitutions of

$$F_i^{(1)}(x) = \frac{f_i(x)}{g_i(x)}$$

and

$$F_i^{(2)}(x) = \frac{l_{i1}}{x} + \frac{l_{i2}}{x^2} + \cdots + \frac{l_{ih_i}}{x^{h_i}},$$

into expressions (4) for F_i, $i = 1, \dots m$, give, for each of these expressions, two rational functions with relatively prime denominators. Thereby, the rational functions are polynomials. By freeing $A(x)$ and $B(x)$ from the factor x (denote the result by $\tilde{A}(x), \tilde{B}(x)$) we compute $q\mathrm{UD}(\tilde{A}(x), \tilde{B}(x))$ and obtain $V(x)$ which is divisible by all $g_i(x)$, $i = 1, \dots, m$.

Now, it is sufficient to find an upper bound H for all $h_1, \dots, h_m$ and then it will be possible to use $U(x) = x^H V(x)$ as a universal denominator for all rational solutions of (1).

As for the second algorithm described in the previous subsection for working with the denominators, we can start with $\tilde{A}(x), \tilde{B}(x)$ and after all the described substitutions, execute one more by using x^H as the denominator of the right-hand side.

To obtain a bound H one can use the technique of indicial equations. (This technique is well-known in the theory of linear ordinary differential equations.) The main computational task connected with the construction of the indicial equation in the differential case is to reduce the given system to the super-irreducible form [16,12]. To the author's knowledge, the super-irreducible form for q-difference systems has not been considered yet. But in any case we can use a universal approach called *EG-eliminations* [2]. This approach allows one, in particular, to construct the indicial equations for differential and q-difference equations.

In other words, any rational Solution (5) of system (1) can be considered as a solution in the class of Laurent series: $F_i^{(1)}(x)$ and $F_i^{(2)}(x)$, $i = 1, \dots, m$, which are, resp., regular and singular parts of the Laurent solution. Thus, an upper bound for its pole order at $x = 0$ can be taken as H. *EG*-eliminations allow one to find such a bound.

The general scheme of using EG-method is the following (see [2] for details). Rewrite system (1) in the operator form

$$\begin{pmatrix} p_1Q - a_{11} & -a_{12} & \dots & -a_{1m} \\ -a_{21} & p_2Q - a_{22} & \dots & -a_{2m} \\ \vdots & \vdots & & \vdots \\ -a_{m1} & -a_{m2} & \dots & p_mQ - a_{mm} \end{pmatrix} y(x) = b(x), \tag{6}$$

where the operator Q is such that $Qf(x) = f(qx)$ for any function $f(x)$. Consider $y_i(x), b_i(x)$, $i = 1,\dots,m$, as Laurent series. Let $z_i(n), c_i(n)$, $i = 1,\dots,m$, be the sequences of coefficients of these series, $-\infty < n < \infty$. Consider the mapping $\mathcal{R}_q : K[x,Q] \to K[q^n, E^{-1}]$ defined by

$$\mathcal{R}_q Q = q^n, \quad \mathcal{R}_q x = E^{-1}, \tag{7}$$

where the operator E is such that $Ef(x) = f(x+1)$ for any function $f(x)$. (In [8] it is shown that $\mathcal{R}_q$ is an isomorphism from $K[x,Q]$ onto $K[q^n,E^{-1}]$.) Applying $\mathcal{R}_q$ to the elements of the operator matrix of (6) we get the operator matrix of the recurrent system for the column of sequences $z(n) = (z_1(n),\dots,z_m(n))^{\mathrm{T}}$. This system can be rewritten in the form

$$P_l z(n+l) + P_{l-1} z(n+l-1) + \cdots + P_t z(n+t) = c(n), \tag{8}$$

where l,t are integers, $l \geqslant t$; $c(n) = (c_1(n),\dots,c_m(n))^{\mathrm{T}}$; $P_l,\dots,P_t$ are $m \times m$-matrices over $K[n]$ with P_l and P_t (the *leading* and *trailing* matrices of the system) being nonzero but the determinants of P_l, P_t can vanish for all n. The process of EG-eliminations in the *explicit* matrix $P = (P_l | P_{l-1} | \dots | P_t)$ allows one to transform (8) into an equivalent system S with nonsingular leading (or analogously, trailing) matrix. Suppose that by using EG-eliminations we constructed the corresponding system S of the form (8) with non-singular $P_l(n)$. Then, $\det P_l(n) = 0$ is the indicial equation for (1) at the point 0. The following theorem can be easily proven in the usual way.

Theorem 1. *Let* $p(n) = \det P_l(n)$ *be a nonzero polynomial in* q^n. *Let* n_0 *be the smallest integer root of* $p(n) = 0$ *if such roots exist and* $n_0 = 1$ *otherwise. Then the pole orders of a Laurent series solution* $y(x) = (y_1(x),\dots,y_m(x))$ *of* (1) *do not exceed* $|\min\{n_0 + l, 0\}|$.

Proof. All $z_i(n)$ are equal to 0 for all negative integer n with large enough $|n|$. Besides, $c_i(n) = 0$ for all negative n, $i = 1,\dots,m$. The matrix $P_l(n)$ is invertible for all $n < n_0$ and we can use recurrence (8) to obtain $z_i(n) = 0$, $i = 1,\dots,m$, for all $n < n_0 + l$. □

3. Polynomial solutions

After performing substitution (2) and clearing the denominators, we need to solve the problem of finding polynomial solutions of a system of form (1). It is sufficient to find an upper bound for the degrees of all $y_1(x),\dots,y_m(x)$. Then, all polynomial solutions of (1) can be found by the method of undetermined coefficients (the problem can be reduced to a system of linear algebraic equations). Less costly approaches such as those proposed for difference system in [4] are also possible.

We can construct the corresponding recurrent system for (1). Suppose that we transformed the recurrent system to the system S of form (8) with non-singular $P_t(n)$ using EG-eliminations. Then $\det P_t(n)=0$ is the indicial equation for (1) at ∞. The following theorem can be easily proven in the usual way.

Theorem 2. *Let $p(n)=\det P_t(n)$ be a nonzero polynomial in q^n. Let n_1 be the largest integer root of $p(n)=0$ if such roots exist and $n_1=-1$ otherwise. Let $d=\max_{i=1}^m \deg b_i$ for (1). Then the degrees of the components of polynomial solution $y(x)=(y_1(x),\dots,y_m(x))$ of (1) do not exceed $\max\{n_1+t, d+t\}$.*

Proof. For polynomial solutions, all $z_i(n)$ are equal to 0 for all large enough n. Besides, $c_i(n)=0$ for all $n>d$, $i=1,\dots,m$. The matrix $P_t(n)$ is invertible for all $n>n_1$ and we can use recurrence (8) to obtain $z_i(n)=0$, $i=1,\dots,m$, for all $n>\max\{n_1+t,d+t\}$. □

4. Example

Let us consider the following system of q-difference equations:

$$(-qx+q^3x)\,y_1(qx)+(x-q^4x)y_1(x)+(q^4x+100q^4-q^2x-100q^2)y_2(x)=0$$

$$(qx+100)y_2(qx)-xy_1(x)=0.$$

Here

$$p_1(x)=-qx+q^3x,\quad p_2(x)=xq+100$$

and the corresponding matrix (3) is

$$\begin{pmatrix} x-q^4x & q^4x+100q^4-q^2x-100q^2 \\ -x & 0 \end{pmatrix}.$$

The inverse matrix is

$$\begin{pmatrix} 0 & -\dfrac{1}{x} \\ \dfrac{1}{q^2(q^2x+100q^2-x-100)} & -\dfrac{q^2+1}{(x+100)q^2} \end{pmatrix}.$$

Thus,

$$A(x)=q^4x^2+100q^4x-q^2x^2-100q^2x,\quad B(x)=-q^2x^2-100xq+q^4x^2+100q^3x.$$

Freeing them from the factor x

$$\tilde{A}(x)=q^2(q^2x+100q^2-x-100), \quad \tilde{B}(x)=q(q^3x-xq-100+100q^2)$$

we obtain $V=q\mathrm{UD}(\tilde{A}(x),\tilde{B}(x))=x+100$.

Now, we should find the exponent of x in the universal denominator. The corresponding recurrent system has the explicit matrix

$$\begin{pmatrix} 0 & q^2(100q^2-100) & q^{n+2}-q^n-q^4+1 & q^2(q^2-1) \\ 0 & 100q^n & -1 & q^n \end{pmatrix},$$

with $l=0$, $t=-1$. EG-eliminations lead to the system with the following leading matrix:

$$\begin{pmatrix} 100(q^{2n+4}-q^{2n+2}+q^{n+1}-q^{n+5}+q^4-q^2) & 0 \\ 0 & 100(q^4-q^2) \end{pmatrix}.$$

The equation $10000(q^{2n+4}-q^{2n+2}+q^{n+1}-q^{n+5}+q^4-q^2)(q^4-q^2)=0$ has the integer roots $-1, 1$. So the degree of the pole is $\leqslant |-1+l|=1$ and the universal denominator is

$$U(x)=xV(x)=x(x+100).$$

After the substitution of the computed universal denominator, the system is

$$((q^2-1)x^2+100(q^2-1)x)z_1(qx)-(q(q^4+1)x^2+100(1-q^4)x)z_1(x)$$
$$-(q^3(1+q^2)x^2-100q^2((q^3+q^2-1)x+100q^2-100)z_2(x)=0,$$
$$(x+100)z_2(qx)-qxz_1(x)=0.$$

The corresponding recurrent system has the explicit matrix

$$\begin{pmatrix} 0 & 0 \\ q^2(10000q^2-10000) & 100q^n \\ 100q^{n+1}-100q^{n-1}-100q^4+100 & -q \\ q^2(100q^3+100q^2-100q-100) & q^{n-1} \\ -q^5+q+q^n-q^{n-2} & 0 \\ q^2(q^3-q) & 0 \end{pmatrix}^{\mathrm{T}}$$

with $l=0$, $t=-2$. EG-eliminations lead to the system with the following trailing matrix:

$$\begin{pmatrix} 0 & q^{2n-2}-q^{2n-4}-q^{n+3}+q^{n-1}+q^6-q^4 \\ -q & q^{n-2} \end{pmatrix}.$$

The polynomial $-q(q^{2n-2}-q^{2n-4}-q^{n+3}+q^{n-1}+q^6-q^4)$ has the integer roots 3, 5. So, the polynomial solutions of the system have degree $\leqslant 5+t=3$. It leads us to the following solution of the system for finding the numerators

$$\left[100\_c_1 + x\_c_1 + x^2\_c_2 + \frac{x^3\_c_2}{100},\ x\_c_1 + \frac{1}{100}\frac{x^3\_c_2}{q^2}\right]$$

and correspondingly to the solution of the given system

$$\left[\frac{x^2\_c_2 + 100\_c_1}{x},\ \frac{100\_c_1 q^2 + x^2\_c_2}{(x+100)q^2}\right].$$

5. Implementation

This algorithm is implemented in Maple V by D.Khmelnov.

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