LINEAR PROGRAMMING PROJECTION ALGORITHMS

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Large-scale LP (linear programming) problems usually have more than one solution. Techniques such as the simplex methods and interior point methods make it possible to obtain different solutions in the case of nonuniqueness. For example, the simplex method yields a solution belonging to a vertex of polyhedron. Some variants of the interior point method converge to a solution satisfying the strict complementary slackness condition.

LP projection method is close to the quadratic penalty function method and to the modified Lagrangian function method. This method yields the exact projection of a given point on the solution set of primal LP problem as a result of the single unconstrained maximization of an auxiliary piecewise quadratic concave function for any sufficiently large values of the penalty parameter. A generalized Newton method with a stepsize chosen using Armijo's rule was used for unconstrained maximization. The proof of globally convergent finitely terminating generalized Newton method for piecewise quadratic function was given in Mangasarian [1] and Kanzow *et al.* [2]. LP projection method solves LP problems with a very large ($\approx 10^7$) number of variables and moderate ($\approx 10^5$) number of constraints.

In a similar way, the exact projection of a given point on the solution set of the dual LP problem can be obtained by nonnegative constrained maximization of auxiliary quadratic function for sufficiently large but finite values of the penalty parameter.

Consider the primal linear program in the standard form

$$f_* = \min_{x \in X} c^\top x, X = \{x \in R^n : Ax = b, x \geq 0_n\} \qquad (P)$$

together with its dual

$$f_* = \max_{u \in U} b^\top u, U = \{u \in R^m : A^\top u \leq c\}, \qquad (D)$$

where $A \in R^{m \times n}$, $c \in R^n$, and $b \in R^m$ are given, x is a primal variable and u is a dual variable, 0_i denotes the i-dimensional zero vector.

Everywhere we assume that the solution set X_* of the primal problem (P) is nonempty, and hence the solution set U_* of the dual problem (D) is also nonempty.

Consider the problems of finding the least 2-norm projection $\hat{x}_*$ of the point $\hat{x}$ on the solution set X_* and the least 2-norm projection $\hat{u}_*$ of the point $\hat{u}$ on the solution set U_*:

$$\frac{1}{2}\|\hat{x}_* - \hat{x}\|^2 = \min_{x \in X_*} \frac{1}{2}\|x - \hat{x}\|^2,$$
$$X_* = \left\{x \in R^n : Ax = b, c^\top x = f_*, x \geq 0_n\right\}, \qquad (1)$$

$$\frac{1}{2}\|\hat{u}_* - \hat{u}\|^2 = \min_{u \in U_*} \frac{1}{2}\|u - \hat{u}\|^2,$$
$$U_* = \left\{u \in R^m : A^\top u \leq c, b^\top u = f_*\right\}. \qquad (2)$$

Here, the Euclidian norm of vectors is used, and f_* is an *a priori* unknown optimal value of the objective function of the original LP problems (P) and (D).

Let us introduce the Lagrange functions for these problems:

$$L^1(x, p, \beta) = \frac{1}{2}\|x - \hat{x}\|^2 + p^\top(b - Ax) + \beta(c^\top x - f_*),$$

$$L^2(u, y, \alpha) = \frac{1}{2}\|u - \hat{u}\|^2 + y^\top(A^\top u - c) + \alpha(f_* - b^\top u).$$

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Here, $p \in R^m$, $\beta \in R^1$, and $y \in R^n_+$, $\alpha \in R^1$ are Lagrange multipliers for problems (1) and (2), respectively.

The problems dual to problems (1) and (2) have the forms

$$\max_{p\in R^m}\max_{\beta\in R^1}\min_{x\in R^n_+} L^1(x,p,\beta), \tag{3}$$

$$\max_{y\in R^n_+}\max_{\alpha\in R^1}\min_{u\in R^m} L^2(u,y,\alpha). \tag{4}$$

The solutions of the inner minimization problems in Equations (3) and (4) have the following forms, respectively:

$$x = (\hat{x} - A^\top p - \beta c)_+, \tag{5}$$

$$u = \hat{u} + \alpha b - Ay, \tag{6}$$

where a_+ denotes the vector a with all negative components replaced by zeros.

Substituting Equation (5) into the Lagrange function $L^1(x,p,\beta)$ and Equation (6) into the Lagrange function $L^2(u,y,\alpha)$, we obtain the dual functions of problems (3) and (6), respectively:

$$\begin{aligned}\hat{L}^1(p,\beta) &= b^\top p - \frac{1}{2}\|(\hat{x} + A^\top p - \beta c)_+\|^2 \\ &\quad - \beta f_* + \frac{1}{2}\|\hat{x}\|^2, \\ \hat{L}^2(y,\alpha) &= -c^\top y - \hat{u}^\top(\alpha b - Ay) \\ &\quad - \frac{1}{2}\|\alpha b - Ay\|^2 + \alpha f_*.\end{aligned}$$

The function $\hat{L}^1(p,\beta)$ is concave, piecewise quadratic, and continuously differentiable in variables p and β. The function $\hat{L}^2(y,\alpha)$ is concave, quadratic, and twice continuously differentiable in both variables.

Dual problems (3) and (4) are reduced to the outer maximization problems, respectively:

$$\max_{p\in R^m\,\beta\in R^1} \hat{L}^1(p,\beta), \tag{7}$$

$$\max_{y\in R^n_+\,\alpha\in R^1} \hat{L}^2(y,\alpha). \tag{8}$$

Solving problem (7), we find optimal p and β. Substituting them into Equation (5), we obtain $\hat{x}_*$, which is the projection of the point $\hat{x}$ on the solution set of the primal LP problem (P).

Solving problem (8), we find optimal y and α. Substituting them into Equation (6), we obtain $\hat{u}_*$, which is the projection of the point $\hat{u}$ on the solution set of the dual LP problem (D).

Unfortunately, maximization problems (7) and (8) contain an *a priori* unknown value f_*, which is the optimal value of the objective function of the original LP problem. However, these problems can be simplified by eliminating this difficulty. For this purpose, the following simplified unconstrained maximization problem is solved instead of problem (7):

$$\begin{aligned}&\max_{p\in R^m} S^1(p,\beta,\hat{x}), \text{where } S^1(p,\beta,\hat{x}) \\ &\quad = b^\top p - \frac{1}{2}\|(\hat{x} + A^\top p - \beta c)_+\|^2.\end{aligned} \tag{9}$$

Here the scalar β is fixed.

Instead of problem (8), the following simplified maximization problem on the positive orthant is solved:

$$\begin{aligned}&\max_{y\in R^n_+} S^2(y,\alpha,\hat{u}), \text{where } S^2(y,\alpha,\hat{u}) \\ &\quad = -c^\top y + \hat{u}^\top Ay - \frac{1}{2}\|\alpha b - Ay\|^2.\end{aligned} \tag{10}$$

Here the scalar α is also fixed.

Let us first consider problem (9) and its relation to the primal LP problem (P). Note that, in contrast to problem (1), dual problem (7) has many solutions. Naturally, the question that arises is of finding the minimal value β_* of the Lagrange multiplier β among all solutions of problem (7). Once such β_* is found, one can fix $\beta \ge \beta_*$ in dual problem (7) and maximize the dual function $\hat{L}^1(p,\beta)$ only with respect to variable p, that is, solve problem (9). In this case, the pair $[p,\beta]$ is a solution to problem (7), and the triplet $[\hat{x}_*, p, \beta]$ is a saddle point of problem (1), where the projection $\hat{x}_*$ of the point $\hat{x}$ on the solution set X_* is defined by problem (5).

Theorem 1 **[3].** *There exists β_* such that for all $\beta \geq \beta_*$ the unique least 2-norm projection $\hat{x}_*$ of a point $\hat{x}$ on X_* is given by*

$$\hat{x}_* = (\hat{x} + A^\top p(\beta) - \beta c)_+,$$

where $p(\beta)$ is a maximizer of $S^1(p, \beta, \hat{x})$ in problem (9).

Theorem 1 makes it possible to replace problem (7), which contains an *a priori* unknown value f_*, by problem (9), which involves the half-interval $[\beta_*, +\infty]$ instead of this value. This essentially simplifies the calculations. Note that the value β_* may be negative. This occurs when the projection of the point $\hat{x}$ on the solution set X_* coincides with the projection of this point on the feasible set X. The estimation of the threshold value β_* is given in Golikov and Evtushenko [3,5].

The next theorem tells us that a solution to problem (D) can be obtained from the single unconstrained maximization problem (9) if a point $\hat{x}_* \in X_*$ is found according to Theorem 1.

Theorem 2 **[3].** *For all $\beta > 0$ and all $\hat{x} = x_* \in X_*$ an exact solution to dual problem (D) is given by $u_* = p(\beta)/\beta$, where $p(\beta)$ is a solution to the unconstrained maximization problem (9).*

To solve the primal and dual LP problems simultaneously, one can use the following iterative process:

$$p_{s+1} \in \arg\max_{p \in R^m} \left\{ b^\top p - \frac{1}{2} \|(x_s + A^\top p - \beta c)_+\|^2 \right\} \tag{11}$$

$$x_{s+1} = (x_s + A^\top p_{s+1} - \beta c)_+, \tag{12}$$

where x_0 is an arbitrary starting point.

Theorem 3 **[3].** *For all $\beta > 0$ and for arbitrary starting point x_0, the iterative process (11)–(12) converges to $x_* \in X_*$ in a finite number of iterations ω. The formula $u_* = p_{\omega+1}/\beta$ provides an exact solution to the dual problem (D).*

Iterative process (11,12) is finite. It gives an exact solution to the primal problem (P) and the exact solution to the dual problem (D). Note that one can use this method even when one is unaware of the threshold value β_*. However, if the chosen coefficient is less than the threshold value, then this method, in a finite number of steps, yields some solution x_* to the primal problem (P) which is not a projection of the starting point x_0 on the solution set X_*. Note that $x_\omega = x_* \in X_*$ is the projection of the point $x_{\omega-1}$ on X_*.

The subsequent theorems are similar to Theorems 1–3.

Theorem 4 **[4].** *There exists α_* such that for all $\alpha \geq \alpha_*$ the unique least 2-norm projection $\hat{u}_*$ of a point $\hat{u}$ on U_* is given by*

$$\hat{u}_* = \hat{u} + \alpha b - Ay(\alpha),$$

where $y(\alpha)$ is a point maximizing $S^2(y, \alpha, \hat{u})$ on R^n_+.

The estimation of the threshold value β_* is given in Evtushenko *et al*. [4].

Theorem 5 **[4].** *For all $\alpha > 0$ and all $\hat{u} = u_* \in U_*$ an exact solution to primal problem (P) is given by $u_* = y(\alpha)/\alpha$, where $y(\alpha)$ is a point maximizing $S^2(y, \alpha, u_*)$ on R^n_+.*

To solve the primal and dual LP problems simultaneously, one can use the following iterative process:

$$y_{s+1} \in \arg\max_{y \in R^n_+} \left\{ -c^\top y + u_s^\top Ay - \tfrac{1}{2} \|\alpha b - Ay\|^2 \right\}, \tag{13}$$

$$u_{s+1} = u_s + \alpha b - Ay_{s+1}. \tag{14}$$

Theorem 6 **[4].** *For all $\alpha > 0$ and for arbitrary starting point u_0, the iterative process (13)–(14) converges to $u_* \in U_*$ in a finite number of iterations ν. The formula $x_* = y_{\nu+1}/\alpha$ provides an exact solution to primal LP problem (P).*

Note that $u_* \in U_*$ is the projection of the point $u_{\nu-1}$ on the solution set U_* of the problem (D).

The unconstrained maximization in Equations (9) and (11) can be performed by any method, for example, by the conjugate gradient method. However, the generalized Newton method [1] turns out to be much more efficient.

The objective function $S^1(p, \beta, \hat{x})$ of problems (9) is concave, piecewise quadratic, and differentiable. The ordinary Hessian does not exist for this function because the gradient

$$\frac{\partial}{\partial p} S^1(p, \beta, \hat{x}) = b - A(\hat{x} + A^\top p - \beta c)_+.$$

is not differentiable. However, for this function, one can define the generalized Hessian matrix, which is an $m \times m$ symmetric negative semidefinite matrix of the form

$$\frac{\partial^2}{\partial p^2} S^1(p, \beta, \hat{x}) = -AD(z)A^\top.$$

Here, $D(z)$ denotes the $n \times n$ diagonal matrix whose ith diagonal entry z^i is equal to 1 if $(\hat{x} + A^\top p - \beta c)^i > 0$ and z^i is equal to zero if $(\hat{x} + A^\top p - \beta c)^i \leq 0$ $(i = 1, 2, \ldots, n)$. The generalized Hessian matrix may be singular. Therefore, we use the following generalized Newton method:

$$p_{s+1} = p_s - \mu \left[\frac{\partial^2}{\partial p^2} S^1(p, \beta, \hat{x}) - \delta I_m \right]^{-1} S_p(p, \beta, \hat{x}),$$

where δ is a small positive number (typically $\delta = 10^{-4}$), I_m is the identity matrix of size $m \times m$, and μ is a stepsize chosen using Armijo's rule.

Unfortunately the generalized Newton method cannot be applied to the nonnegativity constrained maximization problem (10) directly. By incorporating the nonnegativity constraint $y \geq 0_n$ into the objective function of problem (10) as a penalty term, we have the unconstrained maximization problem

$$\max_{y \in R^n} \left\{ -c^\top y + \hat{u}^\top Ay - \frac{1}{2}\|\alpha b - Ay\|^2 - \frac{\tau}{2}\|(-y)_+\|^2 \right\},$$

where $\tau > 0$ is a penalty parameter. In this case, we obtained the optimal solution y only in limit as $\tau \to +\infty$. This property complicates the computation. There exists a very simple way to overcome this shortcoming.

Let a vector $w \in R^{m+n}$ consist of two vectors $w^\top = [u^\top, v^\top]$, where $u \in R^m$, $v \in R^n$. Consider the following LP problem

$$f_* = \max_{w \in W} b^\top u,$$
$$W = \{u \in R^m, v \in R^n : A^\top u + v = c, v \geq 0_n\}, \tag{D'}$$

which is equivalent to dual problem (D). The solution set of this problem is denoted by $W_* = [U_* \times V_*]$. For a given point $\hat{w}$ we find the least 2-norm projection $\hat{w}_*$ on W_* as a solution to the following minimization problem:

$$\frac{1}{2}\|\hat{u}_* - \hat{u}\|^2 + \frac{1}{2}\|\hat{v}_* - \hat{v}\|^2 = \min_{w \in W_*} \left\{ \frac{1}{2}\|u - \hat{u}\|^2 + \frac{1}{2}\|v - \hat{v}\|^2 \right\},$$
$$W_* = \{u \in R^m, v \in R^n : A^\top u + v = c, v \geq 0_n, b^\top u = f_*\}.$$

Using an approach similar to that used above we arrive at the following unconstrained maximization problem:

$$\max_{y \in R^n} S^3(y, \gamma, \hat{w}), \text{where } S^3(y, \gamma, \hat{w}) = -c^\top y + \hat{u}^\top Ay - \frac{1}{2}\|\gamma b - Ay\|^2 - \frac{1}{2}\|(\hat{v} - y)_+\|^2. \tag{15}$$

The following theorems hold.

Theorem 7. *There exists γ_* such that for all $\gamma \geq \gamma_*$ the unique least 2-norm projection $\hat{w}_*^\top = [\hat{u}_*^\top, \hat{v}_*^\top]$ of a point $\hat{w}^\top = [\hat{u}^\top, \hat{v}^\top]$ on W_* is given by*

$$\hat{u}_* = \hat{u} + \gamma b - Ay(\gamma),$$
$$\hat{v}_* = (\hat{v} - y(\gamma))_+,$$

where $y(\gamma)$ is a solution to the unconstrained maximization problem (15).

Table 1. Comparative Results

$m \times n \times \rho$	Solver	T (s)	Iterations	Δ_1	Δ_2	Δ_3
	LPP (MATLAB)	55.0	12	1.5×10^{-8}	1.8×10^{-12}	1.2×10^{-7}
	BPMPD (interior point)	37.4	23	2.3×10^{-10}	1.8×10^{-11}	1.1×10^{-10}
$500 \times 10^4 \times 1$	MOSEK (interior point)	87.2	6	9.7×10^{-8}	3.8×10^{-9}	1.6×10^{-6}
	CPLEX (interior point)	80.3	11	1.8×10^{-8}	1.1×10^{-7}	0.0
	CPLEX (simplex)	61.8	8308	8.6×10^{-4}	1.9×10^{-10}	7.2×10^{-3}
	LPP (MATLAB)	155.4	11	6.1×10^{-10}	3.4×10^{-13}	3.6×10^{-8}
	BPMPD (interior point)	223.5	14	4.6×10^{-9}	2.9×10^{-10}	3.9×10^{-9}
$3000 \times 10^4 \times 0.01$	MOSEK (interior point)	42.6	4	3.1×10^{-8}	1.2×10^{-8}	3.7×10^{-8}
	CPLEX (interior point)	69.9	5	1.1×10^{-6}	1.3×10^{-7}	0.0
	CPLEX (simplex)	1764.9	6904	3.0×10^{-3}	8.1×10^{-9}	9.3×10^{-2}
$1000 \times (5 \times 10^6) \times 0.01$	LPP (MATLAB)	1007.5	10	3.9×10^{-8}	1.4×10^{-13}	6.1×10^{-7}
$1000 \times 10^5 \times 1$	LPP (MATLAB)	2660.8	8	2.1×10^{-7}	1.4×10^{-12}	7.1×10^{-7}

Theorem 8. *For all $\gamma > 0$ and $\hat{w} = w_* \in W_*$ an exact solution to primal problem (P) is given by $x_* = y(\gamma)/\gamma$, where $y(\gamma)$ is a solution to the unconstrained problem (15).*

To solve the primal and dual LP problems simultaneously, one can use the following iterative process similarly to the process (13,14):

$$y_{s+1} \in \arg\max_{y \in R^n} \left\{ -c^\top y + u_s^\top Ay - \frac{1}{2}\|\gamma b - Ay\|^2 - \frac{1}{2}\|(v_s - y)_+\|^2 \right\}, \tag{16}$$

$$u_{s+1} = u_s + \gamma b - Ay_{s+1}, \tag{17}$$

$$v_{s+1} = (v_s - y_{s+1})_+. \tag{18}$$

Theorem 9. *For all $\gamma > 0$ and for arbitrary starting point w_0, the iterative process (16–18) converges to $w_* \in W_*$ in a finite number of iterations σ. The formula $x_* = y_{\sigma+1}/\gamma$ gives an exact solution to the primal problem (P).*

The proofs of Theorems 7–9 are similar to the proofs of Theorems 1–3, respectively. The goal function $S^3(y, \gamma, \hat{w})$ of unconstrained maximization problem (15) is piecewise quadratic concave function. Therefore, one can define its generalized Hessian, which is $m \times m$ symmetric negative semidefinite matrix:

$$\frac{\partial^2}{\partial y^2} S^3(y, \gamma, \hat{w}) = -A^\top A - D(z).$$

Here, $D(z)$ denotes the $n \times n$ diagonal matrix with diagonal elements z^i, $i = 1, 2, \ldots, n$. If $(\hat{v} - y)^i > 0$ then $z^i = 1$, if $(\hat{v} - y)^i \leq 0$ then $z^i = 0$. Now, for solving problem (15) one can use generalized Newton method.

There is an important difference between problems (10) and (15). In the first case, we look for the projection of a given point $\hat{u}$ on U_*, and in the second case we project a point $\hat{W} = [\hat{u}, \hat{v}]$ on the solution set W_*. Let $\hat{u}_*^1$ and $\hat{u}_*^2$ denote the projections of point $\hat{u}$ and $[\hat{u}, \hat{v}]$ in the first and second cases, respectively. Then the following inequality holds:

$$\|\hat{u}_*^1 - \hat{u}\| \leq \|\hat{u}_*^2 - \hat{u}\|.$$

The comparison of LP projection methods (which were implemented in MATLAB) with some well-known commercial and research software packages showed that they are competitive with the simplex and the interior point methods [1,5].

Table 1 presents the results of the test computations obtained using the program LPP, which implements method (11,12) in MATLAB, and other commercial and research packages [5]. Four randomly generated LP problems were solved on a 2.0 GHz Celeron computer with 1 Gb of memory. The following packages were used: BPMPD v. 2.3 (the interior point method) [6], MOSEK v.2.0 (the interior point method) [7], and the popular commercial package CPLEX (v.6.0.1, the interior point and simplex methods).

Table 1 shows the dimensions m and n of the problems, the density ρ of the nonzero

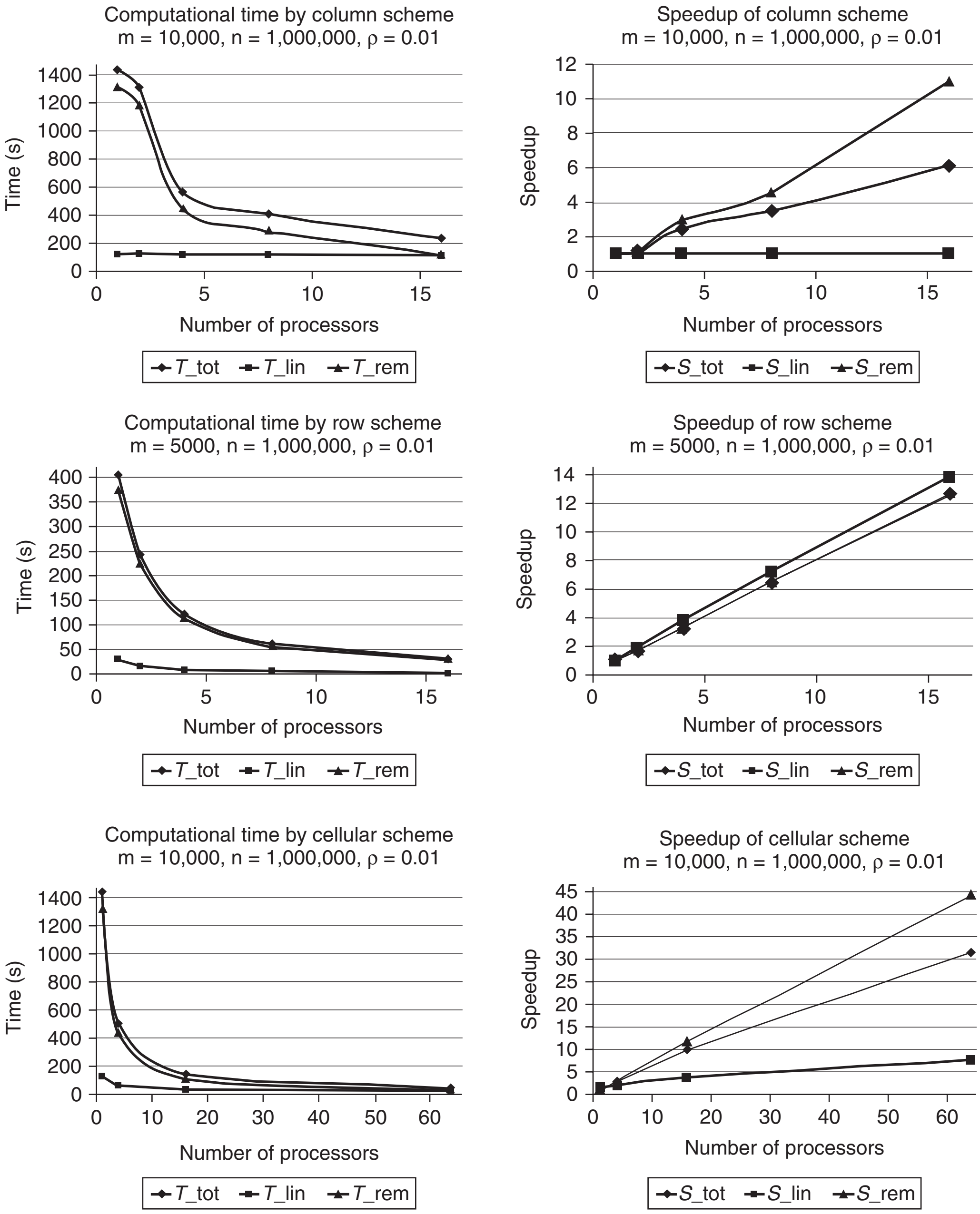

Figure 1. Computational time and speedup diagrams.

entries in the matrix A, the time T needed to solve the the LP in seconds, and the number of iterations (for LP projection method (LPP), the total number of systems of linear equations that were solved in the Newton method when problem (11) was solved). Everywhere, parameter β was equal to 1, which exceeded β_* in these problems. Everywhere, $\hat{x} = 0_n$; that is, the normal solution (projection of the origin onto the primal solution set X_*) was sought in primal LP problem.

The l_∞ norms of the residual vectors were calculated:

$$\Delta_1 = \|Ax - b\|_\infty \le tol,$$
$$\Delta_2 = \|(A^\top u - c)_+\|_\infty \le tol,$$
$$\Delta_3 = |c^\top x - b^\top u| \le tol.$$

It should be stressed that neither the linear program with 5 million variables, $\rho = 0.01$, $m = 1000$, nor those with 10^5 variables, $\rho = 1$, $m = 1000$, could be successfully solved by any of the codes listed above excepting the proposed LPP, which solved these problems in $\approx$16 and $\approx$44 min, respectively, and gave very high accuracy (less than 7.1×10^{-7}). Nevertheless, LPP code did not give the best results for small and moderate-size problems. As is evident from Table 1, BPMPD solver was superior to the others as applied for the first test problem; MOSEK solver showed the best results in the second test problem. Thus, the examples presented above have clearly demonstrated the principal effectiveness of the method proposed in solving the problems with a very large number ($\approx 10^6$) of nonnegative variables and a moderate ($\approx 10^3$) number of equality type constraints.

It is also evident that the results of LP projection algorithm are highly competitive to those of simplex and interior point methods, at the same time outperforming them in solving large LP problems.

Several parallel versions of the generalized Newton method for solving linear programs based on various data decomposition schemes of matrix A (column, row, and cellular schemes) were implemented [8]. The resulting parallel algorithms were successfully used to solve large-scale LP problems (up to several dozens of millions of variables and several hundreds of thousands of constraints) for a relatively dense matrix A. The computational experiments were performed on the cluster consisting of two-processor nodes based on 1.6 GHz Intel Itanium 2 processors connected by Myrinet 2000. For example, for LP problem with 1 million variables and 10,000 constraints, the cellular scheme for 144 processors of the cluster accelerated the computations approximately by a factor of 50, and the computation time was 28 s. LP problem with 2 million variables and 200,000 constraints was solved in about 40 min on 80 processors. Another LP problem with 60 million variables and 4000 constraints was solved by column scheme in 140 s on 128 processors.

The computation time for typical test problems and the speedup for the different data decomposition schemes are shown in Fig. 1; where T_{tot} denotes the computation time in seconds, T_{lin} is the time spent on solving systems of linear equations, T_{rem} is the time spent on the other computations. Furthermore, S_{tot} denotes the parallel speedup in solving LP problem, S_{lin} is the parallel speedup in solving linear systems, and S_{rem} is the speedup of the other computations. Parameter β was equal to 100, which exceeded β_* in these problems. Everywhere, $\hat{x} = 0_n$; that is, the normal solution was sought in primal LP problem.

The development of an efficient solver was found to be a very challenging task; for every parallel variant of the algorithm, a reasonable trade-off between the computational scalability and the scalability in terms of memory had to be found. The highest speedup was obtained for the cellular scheme. However, depending on the dimension of the problem and the sparsity structure of the matrix A, the cellular, the row, or the column scheme was optimal.

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