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Theorems of the Alternative and Their Applications in Numerical Methods

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Revised version

Abstract — New theorems of the alternative are proved for systems of linear equalities and inequalities, which makes it possible to develop several efficient numerical methods. These methods are used to find normal solutions to systems of linear equations and inequalities, to construct separating hyperplanes, to correct inconsistent systems, and to solve nonlinear programming problems. They considerably simplify the implementation of the steepest descent method. The theorems of the alternative proved by Fredholm, Farkas, Gale, Gordan, Stiemke, and others are particular cases of those presented in this paper.

1. INTRODUCTION

There are several theorems of the alternative (see, e.g., [1] – [10]), which were used mainly in proving existence theorems and in deriving extremum conditions in optimization problems. In this paper, we give constructive proofs of new theorems of the alternative, which are designed to construct new computational methods. These proofs make it possible to find normal solutions to systems of linear equations and inequalities, to determine the steepest descent directions in nonlinear programming problems, to construct separating hyperplanes, to correct inconsistent problems, to construct new algorithms for solving linear programming problems, etc.

For a given linear system, an alternative system is constructed in the space whose dimension is equal to the number of equations and inequalities in the original system (not counting constraints on the signs of variables). The original solvable system is solved by minimizing the residuals of the inconsistent alternative system. The results of this minimization are used to find the normal solution (with a minimal Euclidean norm) to the original system. The replacement of the original problem by the minimization of the residuals of the inconsistent alternative system may be advantageous when the dimension of the new variables is less than that of the starting ones. In this case, such a reduction results in the minimization problem in a space of lower dimension and allows one to obtain the normal solution to the original problem.

Many well-known theorems of the alternative (e.g., the Fredholm, Farkas, Gale, Gordan, and Stiemke theorems) follow from the theorems stated in this paper. However, we believe that the principal achievement of this study is that the results obtained offer new opportunities for applying theorems of the alternative in the development of computational methods. A new method for finding normal solutions to systems of linear equations and inequalities based on these theorems is presented in Section 2. These theorems make it possible to considerably simplify the implementation of the steepest descent method (Section 3), to derive simple correction formulas for unsolvable systems (Section 4), to obtain formulas for separating hyperplanes (Section 7), and to develop new methods for solving linear programming problems ([17]).

The proofs of the theorems are based on the duality theory. This work is a sequel to our earlier papers [16] – [19]. The minimization of residuals for proving theorems of the alternative was first suggested in [5]. We also call the reader's attention to [8, 20, 21], where application of theorems of the alternative in constructing numerical methods was discussed.

2. BASIC THEOREMS

Let an $m \times n$ matrix A be given in the form

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

where A_{11}, A_{12}, A_{21}, and A_{22} are rectangular matrices of dimensions $m_1 \times n_1$, $m_1 \times n_2$, $m_2 \times n_1$, and $m_2 \times n_2$, respectively. Let vectors $x \in \mathbb{R}^n$, z, u, and $b \in \mathbb{R}^m$ be represented in partitioned form as $x^\top = [x_1^\top, x_2^\top]$, $z^\top = [z_1^\top, z_2^\top]$, $u^\top = [u_1^\top, u_2^\top]$, and $b^\top = [b_1^\top, b_2^\top]$, where $x_1 \in \mathbb{R}^{n_1}$, $x_2 \in \mathbb{R}^{n_2}$, $n = n_1 + n_2$, $z_1, u_1, b_1 \in \mathbb{R}^{m_1}$, $z_2, u_2, b_2 \in \mathbb{R}^{m_2}$, and $m = m_1 + m_2$. The matrix A and the vector b are assumed to be nonzero ones.

Define the auxiliary sets $\Pi_x = \{[x_1, x_2] : x_1 \in \mathbb{R}^{n_1}_+, x_2 \in \mathbb{R}^{n_2}\}$, $\Pi_z = \{[z_1, z_2] : z_1 \in \mathbb{R}^{m_1}_+, z_2 \in \in \mathbb{R}^{m_2}\}$, and $\Pi_u = \{[u_1, u_2] : u_1 \in \mathbb{R}^{m_1}_+, u_2 \in \mathbb{R}^{m_2}\}$; a vector $w \in \mathbb{R}^{n+1}$ represented as $w^\top = = [w_1^\top, w_2^\top, w_3]$, where $w_1 \in \mathbb{R}^{n_1}$, $w_2 \in \mathbb{R}^{n_2}$, and $w_3 \in \mathbb{R}^1$; and the auxiliary set $\Pi_w = = \{[w_1, w_2, w_3] : w_1 \in \mathbb{R}^{n_1}_+,\ w_2 \in \mathbb{R}^{n_2},\ w_3 \in \mathbb{R}^1\}$. By $\|a\|$ and $\|a\|_1 = \sum_{i=1}^{n} |a^i|$, we denote the Euclidean and first norms of a vector $a \in \mathbb{R}^n$, respectively.

Consider the system of linear equations and inequalities

$$A_{11}x_1 + A_{12}x_2 \geq b_1, \quad A_{21}x_1 + A_{22}x_2 = b_2, \quad x_1 \geq 0_{n_1}. \tag{I}$$

This system can be viewed as a feasible set in the linear programming problem of minimization of $c^\top x$, where $c = 0_n$. For this problem, the dual problem is the maximization of $b^\top u$ on the feasible set defined by the system

$$A_{11}^\top z_1 + A_{21}^\top z_2 \leq 0_{n_1}, \quad A_{12}^\top z_1 + A_{22}^\top z_2 = 0_{n_2}, \quad z_1 \geq 0_{m_1}. \tag{I$'$}$$

System (I)$'$ is said to be *adjoint* to (I).

Consider the system

$$A_{11}^\top u_1 + A_{21}^\top u_2 \leq 0_{n_1}, \quad A_{12}^\top u_1 + A_{22}^\top u_2 = 0_{n_2}, \quad b_1^\top u_1 + b_2^\top u_2 = \rho, \quad u_1 \geq 0_{m_1}, \tag{II}$$

where $\rho > 0$ is an arbitrary fixed positive number. The condition $b_1^\top u_1 + b_2^\top u_2 = \rho$ implies that the adjoint system (I)$'$ has no trivial solution.

The adjoint of system (II) is

$$A_{11}w_1 + A_{12}w_2 - b_1 w_3 \geq 0_{m_1}, \quad A_{21}w_1 + A_{22}w_2 - b_2 w_3 = 0_{m_2}, \quad w_1 \geq 0_{n_1}. \tag{II$'$}$$

We denote the sets of solutions to (I), (I)$'$, (II), and (II)$'$ by X, Z, U, and W, respectively. If systems (I) or (II) are solvable, we write $X \neq \emptyset$ or $U \neq \emptyset$, respectively. Unlike systems (I) and (II), the adjoint systems (I)$'$ and (II)$'$ always have solutions, since $0_m \in Z$ and $0_{n+1} \in W$. It follows from the form of system (II) that, if it is solvable (unsolvable) for a certain $\rho = \rho_1 > 0$, then it is solvable (unsolvable) for any other $\rho = \rho_2 > 0$.

Let $\operatorname{pen}(x, X)$ denote the penalty for the violation of the condition $x \in X$ calculated at a point $x \in \Pi_x$. By the definition of the penalty function, $x \in \Pi_x$, then $\operatorname{pen}(x, X) = 0$ if and only if $x \in X$. The quantity $\operatorname{pen}(u, U)$ is introduced by analogy. If $u \in \Pi_u$, then $\operatorname{pen}(u, U) = 0$ if and only if $u \in U$.

The penalties are calculated as the Euclidean norms of the residual vectors for systems (I) and (II):

$$\begin{aligned}\mathrm{pen}\,(x, X) &= \left[\|(b_1 - A_{11}x_1 - A_{12}x_2)_+\|^2 + \|b_2 - A_{21}x_1 - A_{22}x_2\|^2\right]^{1/2},\\ \mathrm{pen}\,(u, U) &= \left[\|(A_{11}^\top u_1 + A_{21}^\top u_2)_+\|^2 + \|A_{12}^\top u_1 + A_{22}^\top u_2\|^2 + (\rho - b_1^\top u_1 - b_2^\top u_2)^2\right]^{1/2}.\end{aligned}$$

Here, a_+ is the nonnegative part of the vector a; i.e., the ith component of the vector a_+ is equal to that of the vector a if the latter is nonnegative and is zero otherwise.

To find out whether a system is solvable and, if it is, to solve it, we apply methods of unconstrained minimization to either of the following problems:

$$I_1 = \min_{x \in \Pi_x} \left[\mathrm{pen}\,(x, X)\right]^2/2, \tag{1}$$

$$I_2 = \min_{u \in \Pi_u} \left[\mathrm{pen}\,(u, U)\right]^2/2. \tag{2}$$

In the strict sense, (1) and (2) are not unconstrained minimization problems, since they involve constraints on the signs of the components of x_1 and u_1. However, since most unconstrained minimization methods can easily be modified to allow for constraints on the signs of variables, we will keep this term for problems (1) and (2). Problems (1) and (2) are always solvable, since quadratic objective functions defined on nonempty feasible sets Π_x and Π_u are bounded by zero from below.

Two systems are said to be *mutually alternative* if only one of them is consistent. The solvability or unsolvability of a system can be characterized by a scalar quantity called the minimum residual, which is found by solving problem (1) or (2). Therefore, solution of both problems makes it possible to determine whether the systems are mutually alternative. Let $x^* \in \Pi_x$ and $u^* \in \Pi_u$ be arbitrary solutions of problems (1) and (2), respectively; i.e., $I_1 = [\mathrm{pen}\,(x^*, X)]^2/2$ and $I_2 = [\mathrm{pen}\,(u^*, U)]^2/2$. Then, the following lemma is valid for systems (I) and (II).

Lemma 1 (criterion for alternativity). *Systems* (I) *and* (II) *are mutually alternative if and only if*

$$\mathrm{pen}\,(x^*, X)\,\mathrm{pen}\,(u^*, U) = 0, \quad \mathrm{pen}\,(x^*, X) + \mathrm{pen}\,(u^*, U) > 0. \tag{3}$$

Proof. The former equation in (3) implies that at least one of the systems, (I) or (II), is solvable. The latter equation implies that at least one of the systems is unsolvable. Then, it follows that systems (I) and (II) are mutually alternative. □

Lemma 2. *Systems* (I) *and* (II) *are not solvable simultaneously.*

Proof. Assume the contrary. Let there exist solutions x^* and u^* to systems (I) and (II), respectively. Then, substituting x^* into (I), multiplying the inequality in (I) by u_1^* and the equation (II) by u_2^*, adding the results, and performing simple calculations, we obtain

$$x_1^{*\top}(A_{11}^\top u_1^* + A_{21}^\top u_2^*) + x_2^{*\top}(A_{12}^\top u_1^* + A_{22}^\top u_2^*) \geq b_1^\top u_1^* + b_2^\top u_2^*.$$

By virtue of (II), the left-hand side of this inequality is nonpositive, whereas its right-hand side is strictly positive, since $A_{11}^\top u_1^* + A_{21}^\top u_2^* \leq 0_{n_1}$ and $b_1^\top u_1^* + b_2^\top u_2^* = \rho > 0$. Thus, we arrive at a contradiction. Therefore, systems (I) and (II) cannot be consistent simultaneously and meet the second condition in (3). The lemma is proved. □

The former condition in (3) is proved in Theorem 3 below. The original system (I) is alternative to the alternative system (II). Indeed, every system alternative to system (II) has the form

$$A_{11}w_1 + A_{12}w_2 - b_1w_3 \geq 0_{m_1}, \quad A_{21}w_1 + A_{22}w_2 - b_2w_3 = 0_{m_2}, \quad \rho w_3 = \rho', \quad w_1 \geq 0_{n_1}, \tag{4}$$

where $\rho' > 0$ is an arbitrary positive number. Hence, $w_3 = \rho'/\rho > 0$ Changing the variables in (4), $x_1 = w_1/w_3$ and $x_2 = w_2/w_3$, we obtain the original system (I).

Consider the following two quadratic programming problems:

$$I_1^d = \max_{z \in Z} \left\{ b^\top z - \|z\|^2/2 \right\}, \tag{5}$$

$$I_2^d = \max_{w \in W} \left\{ \rho w_3 - \|w\|^2/2 \right\}. \tag{6}$$

Unlike systems (I) and (II), which may be consistent or inconsistent, problems (1), (2), (5), and (6) always have solutions. Moreover, problems (5) and (6) have unique solutions, since the feasible sets Z and W in these problems are nonempty, and strictly concave quadratic objective functions are bounded from above (see formula (32) below).

Formally, the unconstrained minimization problems (1) and (2) do not have Lagrange functions, which implies that the corresponding dual problems cannot be constructed directly. Nevertheless, one can introduce additional variables to construct artificial constraints and obtain equivalent nonlinear programming problems for which dual problems are well defined.

Lemma 3. *The unconstrained minimization problems* (1) *and* (2) *are dual to problems* (5) *and* (6)*, respectively. Problems* (1) *and* (2) *reduce to equivalent problems of constrained minimization of quadratic functions, which are dual to problems* (5) *and* (6)*, respectively.*

Proof. The first assertion follows from the conventional representation of dual problems for quadratic programming problems (see, e.g., [6, 12, 13, 14]).

The second assertion is not quite conventional. It is based on a two-step representation of problems (1) and (2). This approach was employed in [14, 15].

Let us introduce a vector of additional variables $y \in \mathbb{R}^m$, $y^\top = [y_1^\top, y_2^\top]$, where $y_1 \in \mathbb{R}^{m_1}$ and $y_2 \in \mathbb{R}^{m_2}$ are given by

$$y_1 = b_1 - A_{11}x_1 - A_{12}x_2, \quad y_2 = b_2 - A_{21}x_1 - A_{22}x_2.$$

Then, problem (1) reduces to the equivalent constrained minimization problem

$$I_1 = \min_{[x,y] \in G} f(y), \tag{7}$$

in which the objective function and the feasible set are

$$f(y) = \|(y_1)_+\|^2/2 + \|y_2\|^2/2,$$

$$G = \{[x, y] : A_{11}x_1 + A_{12}x_2 + y_1 = b_1, \ \ A_{21}x_1 + A_{22}x_2 + y_2 = b_2, \ \ x \in \Pi_x\}.$$

Unlike X, the set G is always nonempty.

For the quadratic programming problem (7), the Lagrange function is given by

$$L(x, y, z) = f(y) + z_1^\top (b_1 - A_{11}x_1 - A_{12}x_2 - y_1) + z_2^\top (b_2 - A_{21}x_1 - A_{22}x_2 - y_2),$$

where $z \in \Pi_z$ is the vector of Lagrange multipliers. The expression for the Lagrange function is transformed into

$$L(x, y, z) = f(y) - x_1^\top (A_{11}^\top z_1 + A_{21}^\top z_2) - x_2^\top (A_{12}^\top z_1 + A_{22}^\top z_2) + z_1^\top (b_1 - y_1) + z_2^\top (b_2 - y_2). \tag{8}$$

Define the dual function

$$F(z) = \min_{x \,\in\, \Pi_x} \ \min_{y \in \mathbb{R}^n} L(x, y, z) \tag{9}$$

and consider the problem of finding

$$\max_{z\in\Pi_z} F(z),$$

which is dual to (7).

The necessary and sufficient optimality conditions for problem (9) are

$$\begin{aligned}
L_{x_1}(x,y,z) &= -A_{11}^\top z_1 - A_{21}^\top z_2 \geq 0_{n_1}, \quad D(x_1)(A_{11}^\top z_1 + A_{21}^\top z_2) = 0_{n_1}, \quad x_1 \geq 0_{n_1}, && (10)\\
L_{x_2}(x,y,z) &= -A_{12}^\top z_1 - A_{22}^\top z_2 = 0_{n_2}, && (11)\\
L_{y_1}(x,y,z) &= (y_1)_+ - z_1 = 0_{m_1}, \qquad L_{y_2}(x,y,z) = y_2 - z_2 = 0_{m_2}. && (12)
\end{aligned}$$

Hereinafter, $D(z)$ denotes the diagonal matrix whose ith diagonal element is the ith component of the vector z.

For $z \in \Pi_z$, it follows from (12) that $z = y$. Substituting this into (8) and assuming that $z \in Z$, we find that, by virtue of definition (9) and conditions (10) and (11), the dual function takes the form $F(z) = b^\top z - \|z\|^2/2$. Thus, we arrive at problem (5), which is dual to (7) and, in a sense, to (1). Hence, the unconstrained minimization problem (1) and the quadratic programming problem (5) can be interpreted as mutually dual. Similarly, problems (2) and (6) are mutually dual. The lemma is proved. □

For the problems introduced above, a duality theorem is valid. It states that the optimal values of the objective functions are equal:

$$I_1 = I_1^d, \qquad I_2 = I_2^d. \tag{13}$$

The projection of a point $\bar{x}$ onto a nonempty closed set X is the point $\bar{x}^* \in X$ nearest to $\bar{x}$, i.e., the point that minimizes the functional

$$J = \min_{x\in X} \|\bar{x} - x\| = \|\bar{x} - \bar{x}^*\|. \tag{14}$$

We write $\bar{x}^* = \mathrm{pr}\,(\bar{x}, X)$ and denote the distance from $\bar{x}$ to X by $\mathrm{dist}\,(\bar{x}, X) = \|\bar{x}^* - \bar{x}\|$.

Theorem 1. *Any solution x^* of problem* (1) *determines a unique solution $z^{*\top} = [z_1^{*\top}, z_2^{*\top}]$ to problem* (5) *as*

$$z_1^* = (b_1 - A_{11}x_1^* - A_{12}x_2^*)_+, \qquad z_2^* = b_2 - A_{21}x_1^* - A_{22}x_2^*, \tag{15}$$

and it holds that

$$\|z^*\|^2 = b^\top z^*, \tag{16}$$

$$z^* \perp Ax^*, \qquad z^* \perp (b - z^*), \tag{17}$$

$$z^* = \mathrm{pr}\,(b, Z), \quad \|z^*\| = \mathrm{pen}\,(x^*, X), \quad \|b - z^*\| = \mathrm{dist}\,(b, Z), \tag{18}$$

$$[\mathrm{pen}\,(x^*, X)]^2 + [\mathrm{dist}\,(b, Z)]^2 = \|b\|^2. \tag{19}$$

Proof. The necessary and sufficient minimum conditions for problem (1) at the point x^* are written as

$$\begin{aligned}
&-A_{11}^\top(b_1 - A_{11}x_1^* - A_{12}x_2^*)_+ - A_{21}^\top(b_2 - A_{21}x_1^* - A_{22}x_2^*) \geq 0_{n_1},\\
&D(x_1^*)\left[A_{11}^\top(b_1 - A_{11}x_1^* - A_{12}x_2^*)_+ + A_{21}^\top(b_2 - A_{21}x_1^* - A_{22}x_2^*)\right] = 0_{n_1}, \quad x_1^* \geq 0_{n_1}, \qquad (20)\\
&A_{12}^\top(b_1 - A_{11}x_1^* - A_{12}x_2^*)_+ + A_{22}^\top(b_2 - A_{21}x_1^* - A_{22}x_2^*) = 0_{n_2}.
\end{aligned}$$

Using

$$z_1^* = (b_1 - A_{11}x_1^* - A_{12}x_2^*)_+, \qquad z_2^* = b_2 - A_{21}x_1^* - A_{22}x_2^* \tag{21}$$

in (20), let us show that $z^{*\top} = [z_1^{*\top}, z_2^{*\top}]$ is a solution to problem (5). Conditions (20) are rewritten as

$$A_{11}^\top z_1^* + A_{21}^\top z_2^* \le 0_{n_1}, \qquad A_{12}^\top z_1^* + A_{22}^\top z_2^* = 0_{n_2}, \tag{22}$$

$$D(x_1^*)(A_{11}^\top z_1^* + A_{21}^\top z_2^*) = 0_{n_1} \qquad x_1^* \ge 0_{n_1}. \tag{23}$$

It follows from (21) and (22) that $z^* \in Z$. Multiplying the former relation in (21) by z_1^* and the latter by z_2^*, we obtain

$$\begin{aligned} \|z_1^*\|^2 &= z_1^{*\top}(b_1 - A_{11}x_1^* - A_{12}x_2^*)_+ = z_1^{*\top}(b_1 - A_{11}x_1^* - A_{12}x_2^*) = \\ &= b_1^\top z_1^* - x^{*\top}A_{11}^\top z_1^* - x_2^{*\top}A_{12}^\top z_1^*, \\ \|z_2^*\|^2 &= z_2^{*\top}(b_2 - A_{21}x_1^* - A_{22}x_2^*) = b_2^\top z_2^* - x_1^{*\top}A_{21}^\top z_2^* - x_2^{*\top}A_{22}^\top z_2^*. \end{aligned}$$

Adding the equations obtained, we find that

$$\begin{aligned} \|z^*\|^2 = \|z_1^*\|^2 + \|z_2^*\|^2 &= b_1^\top z_1^* + b_2^\top z_2^* - x_1^{*\top}(A_{11}^\top z_1^* + A_{21}^\top z_2^*) - x_2^{*\top}(A_{12}^\top z_1^* + A_{22}^\top z_2^*) = \\ &= (b - Ax^*)^\top z^* = b^\top z^*. \end{aligned} \tag{24}$$

In Eq. (24), we used the fact that $x_1^{*\top}(A_{11}^\top z_1^* + A_{21}^\top z_2^*) + x_2^{*\top}(A_{12}^\top z_1^* + A_{22}^\top z_2^*) = x^{*\top}A^\top z^* = 0$ by (22) and (23). Thus, Eq. (16) is proved. Moreover, it is proved that $z^{*\top}Ax^* = 0$; i.e., the vectors z^* and Ax^* are orthogonal. Equation (16) can be rewritten as $z^{*\top}(z^* - b) = 0$. The validity of the former relation in (17) follows from this equation.

Let us define the Lagrange function for problem (5) as

$$L(z, x) = b^\top z - \|z\|^2/2 - x_1^\top(A_{11}^\top z_1 + A_{21}^\top z_2) - x_2^\top(A_{12}^\top z_1 + A_{22}^\top z_2) \tag{25}$$

and write out the Kuhn–Tucker conditions:

$$L_{z_1}(z, x) = b_1 - z_1 - A_{11}x_1 - A_{12}x_2 \le 0_{m_1}, \tag{26}$$

$$D(z_1)(b_1 - z_1 - A_{11}x_1 - A_{12}x_2) = 0_{m_1}, \qquad z_1 \ge 0_{m_1}, \tag{27}$$

$$L_{z_2}(z, x) = b_2 - z_2 - A_{21}x_1 - A_{22}x_2 = 0_{m_2}, \tag{28}$$

$$L_{x_1}(z, x) = -(A_{11}^\top z_1 + A_{21}^\top z_2) \ge 0_{n_1}, \quad x_1 \ge 0_{n_1}, \quad D(x_1)(A_{11}^\top z_1 + A_{21}^\top z_2) = 0_{n_1}, \tag{29}$$

$$L_{x_2}(z, x) = -(A_{12}^\top z_1 + A_{22}^\top z_2) = 0_{n_2}. \tag{30}$$

Let us compare the necessary and sufficient minimum conditions (21) – (23) for problem (1) with those for the quadratic programming problem (5) (conditions (26) – (30)). If we substitute x^* and z^* defined by (21) for x and z, respectively, under the Kuhn–Tucker conditions (26) – (30), then (29) and (30) become (22) and (23), respectively. It is easy to see that Eq. (21) ensures the fulfillment of conditions (26) – (28). Thus, the saddle point of the Lagrange function (25) is $[z^*, x^*]$, where z^* and x^* are solutions to problems (5) and (1), respectively. Note that the last two vectors are related by Eq. (15).

By virtue of (5), (13), and (16), we have

$$I_1 = I_1^d = \|z^*\|^2/2 = [\mathrm{pen}\,(x^*, X)]^2/2, \tag{31}$$

which proves the second assertion in (18).

Transforming problem (5)into the equivalent problems

$$I_1^d = \max_{z\in Z} \left[-\|b - z\|^2 + \|b\|^2/2\right] = \|b\|^2/2 - \min_{z\in Z} \|b - z\|^2/2 \tag{32}$$

and using the fact that z^* is the unique solution to (5), we find that $z^* = \text{pr}\,(b, Z)$ and $\|b-z^*\| = \text{dist}\,(b, Z)$. Thus, all assertions in (18) are proved. Using (16), we obtain $\|z^*\|^2 + \|b - z^*\|^2 = \|b\|^2$, which proves (19). The theorem is proved. $\square$

Assertion (16) in Theorem 1 follows from the duality of problems (1) and (5) (see (13)). Equations (15) can be used to express an optimal vector z^* for problem (5) in terms of the optimal vector x^* for problem (1). The vector z^* defined by (15) is called the *minimum residual vector*. Equation (31) leads to the following criterion.

Criterion 1. *System* (I) *is solvable if and only if the minimum residual vector* z^* *is zero* (*problem* (5) *has a zero solution*).

The vector x^* satisfying the necessary and sufficient optimality conditions (20) for problem (1) is said to be a *pseudosolution to system* (I). If $x^* \in X$, then x^* is a solution to system (I). Note that this terminology is used in the least square method. Similarly, solution x^* to problem (2) is referred to as a *pseudosolution to system* (II).

The analysis of problems (2) and (6) is analogous to that of problems (1) and (5), but is somewhat different. The following theorem is the analogue of Theorem 1 that applies to problems (2) and (6). Let $\hat{A} = [-A, b]$ be an $m \times (n + 1)$ matrix and $r \in \mathbb{R}^{n+1}$ be a vector of the form $r^\top = [0_n, \rho]$.

Theorem 2. *Let* $u^{*\top} = [u_1^{*\top}, u_2^{*\top}]$ *be an arbitrary solution to problem* (2)*. Then, a solution* $w^{*\top} = [w_1^{*\top}, w_2^{*\top}, w_3^*]$ *to problem* (6) *can be expressed in terms of* u^* *as*

$$w_1^* = (A_{11}^\top u_1^* + A_{21}^\top u_2^*)_+, \quad w_2^* = A_{12}^\top u_1^* + A_{22}^\top u_2^*, \quad w_3^* = \rho - b_1^\top u_1^* - b_2^\top u_2^* \tag{33}$$

and satisfies the following relations:

$$\|w^*\|^2 = \rho w_3^*, \tag{34}$$

$$w^* \perp \hat{A}^\top u^*, \quad w^* \perp (r - w^*), \tag{35}$$

$$w^* = \text{pr}\,(r, W), \quad \|w^*\| = \text{pen}\,(u^*, U), \quad \|r - w^*\| = \text{dist}\,(r, W), \tag{36}$$

$$[\text{pen}\,(u^*, U)]^2 + [\text{dist}\,(r, W)]^2 = \|r\|^2, \tag{37}$$

$$\|w^*\| \le \rho, \quad 0 \le w_3^* \le \rho, \quad \|w_1^*\|^2 + \|w_2^*\|^2 \le \rho^2/4. \tag{38}$$

Proof. The strict convex quadratic programming problem (6) is the problem of finding the projection of the vector $[0_n^\top, \rho]$ onto the nonempty set W specified by system (II)$'$ of linear equations and inequalities. This problem always has a unique solution. For this problem, there exists a vector of Lagrange multipliers $u \in \mathbb{R}^n$, and the corresponding Lagrange function is

$$L(w, u) = \rho w_3 - \|w\|^2/2 - u_1^\top(b_1 w_3 - A_{11} w_1 - A_{12} w_2) - u_2^\top(b_2 w_3 - A_{21} w_1 - A_{22} w_2).$$

The necessary and sufficient optimality conditions (Kuhn–Tucker conditions) for problem (6) calculated at the saddle point $[w^*, u^*]$ are

$$-w_1^* + A_{11}^\top u_1^* + A_{21}^\top u_2^* \le 0_{n_1}, \quad D(w_1^*)(-w_1^* + A_{11}^\top u_1^* + A_{21}^\top u_2^*) = 0_{n_1}, \quad w_1^* \ge 0_{n_1}, \tag{39}$$

$$-w_2^* + A_{12}^\top u_1^* + A_{22}^\top u_2^* = 0_{n_2}, \tag{40}$$

$$\rho - w_3^* - b_1^\top u_1^* - b_2^\top u_2^* = 0, \tag{41}$$

$$A_{11}w_1^* + A_{12}w_2^* - b_1 w_3^* \geq 0_{m_1}, \quad D(u_1^*)(A_{11}w_1^* + A_{12}w_2^* - b_1 w_3^*) = 0_{m_1}, \quad u_1^* \geq 0_{m_1}, \tag{42}$$

$$A_{21}w_1^* + A_{22}w_2^* - b_2 w_3^* = 0_{m_2}. \tag{43}$$

It follows from (39) – (41) that $w^* \in W$ and w^* can be expressed in terms of u^* by formulas (33). Substituting them into (42) and (43), we obtain

$$A_{11}(A_{11}^\top u_1^* + A_{21}^\top u_2^*)_+ + A_{12}(A_{12}^\top u_1^* + A_{22}^\top u_2^*) - b_1(\rho - b_1^\top u_1^* - b_2^\top u_2^*) \geq 0_{m_1}, \quad u_1^* \geq 0_{m_1},$$

$$D(u_1^*)\left[A_{11}(A_{11}^\top u_1^* + A_{21}^\top u_2^*)_+ + A_{12}(A_{12}^\top u_1^* + A_{22}^\top u_2^*) - b_1(\rho - b_1^\top u_1^* - b_2^\top u_2^*)\right] = 0_{m_1},$$

$$A_{21}(A_{11}^\top u_1^* + A_{21}^\top u_2^*)_+ + A_{22}(A_{12}^\top u_1^* + A_{22}^\top u_2^*) - b_2(\rho - b_1^\top u_1^* - b_2^\top u_2^*) = 0_{m_2}.$$

These relations coincide with the necessary and sufficient optimality conditions for problem (2) calculated at the point u^*. Thus, it follows from the Kuhn–Tucker conditions and the optimality conditions for problem (2) that the vectors w^* and u^* at the saddle point $[w^*, u^*]$ solve problems (6) and (2), respectively.

Equating the objective functions of the primal and dual problems (6) and (2) and using relations (33), we obtain (34).

Relations (35) – (37) are proved by analogy with (17) – (19) in Theorem 1.

It follows from (34) that $w_3^* > 0$ if $\|w^*\| \neq 0$. Let us rewrite (34) as the quadratic equation in w_3^*

$$(w_3^*)^2 - \rho w_3^* + \|w_1^*\|^2 + \|w_2^*\|^2 = 0. \tag{44}$$

Since w^* is the projection of r onto W, the first inequality in (38) follows from the equation $\|r\| = \rho$. The second and third inequalities in (38) follow from the nonnegativity of the absolute term $(w_3^*(\rho - w_3^*) = \|w_1^*\|^2 + \|w_2^*\|^2 \geq 0)$ and the determinant of quadratic equation (44), respectively. The theorem is proved. $\square$

Criterion 2. *System* (II) *is solvable* (*unsolvable*) *if and only if problem* (6) *has a zero* (*nonzero*) *solution* w^*.

Theorem 3. *Let* x^* *and* u^* *be arbitrary solutions to problems* (1) *and* (2), *respectively, and let the minimum residual vectors* z^* *and* w^* *be defined by* (15) *and* (33). *Then, the following assertions are valid:*

(1) *systems* (I) *and* (II) *are mutually alternative; i.e., only one of them is solvable;*

(2) *if system* (I) *is inconsistent, then the normal solution* $\tilde{u}^*$ *to system* (II) *and the minimum residual vector* z^* *of system* (I) *are collinear, and*

$$\tilde{u}^* = \rho z^*/\|z^*\|^2, \qquad z^* = \rho \tilde{u}^*/\||\tilde{u}^*\|^2; \tag{45}$$

(3) *if system* (II) *is inconsistent, then the components of the normal solution* $\tilde{x}^{*\top} = [\tilde{x}_1^{*\top}, \tilde{x}_2^{*\top}]$ *to system* (I) *are*

$$\tilde{x}_1^* = w_1^*/w_3^*, \qquad \tilde{x}_2^* = w_2^*/w_3^*. \tag{46}$$

Proof. It follows from Lemma 2 that systems (I) and (II) cannot be consistent simultaneously. Let us show that one of them must be consistent. Consider the two possible cases separately.

If $X = \emptyset$, then $\mathrm{pen}\,(x^*, X) \neq 0$. The vector $z^* \in Z$ defined by (15) is such that $\|z^*\| \neq 0$. Multiplying both sides of the former equation in (45) by b and taking into account (16), we find that $b^\top \tilde{u}^* = \rho$. Then, it follows that $\tilde{u}^* \in U$; hence, $U \neq \emptyset$. Let us show that $\tilde{u}^*$ is the normal solution to system (II), i.e., a solution to the problem

$$\min_{u \in U} \|u\|^2/2. \tag{47}$$

The Lagrange function for problem (47) is written as

$$L(u, \hat{x}) = \|u\|^2/2 + \hat{x}_1^\top (A_{11}^\top u_1 + A_{21}^\top u_2) + \hat{x}_2^\top (A_{12}^\top u_1 + A_{22}^\top u_2) + \hat{x}_3(\rho - b_1^\top u_1 - b_2^\top u_2)$$

and the dual problem is

$$\max_{\hat{x}_1 \in \mathbb{R}^{n_1}_+} \max_{\hat{x}_2 \in \mathbb{R}^{n_2}} \max_{\hat{x}_3 \in \mathbb{R}^1} \left[\rho \hat{x}_3 - \frac{\|(b_1\hat{x}_3 - A_{11}\hat{x}_1 - A_{12}\hat{x}_2)_+\|^2}{2} - \frac{\|b_2\hat{x}_3 - A_{21}\hat{x}_1 - A_{22}\hat{x}_2\|^2}{2} \right]. \tag{48}$$

The Kuhn–Tucker conditions calculated at the saddle point $[u^*, \hat{x}^*]$, where $u^{*\top} = [u_1^{*\top}, u_2^{*\top}]$ solves problem (47) and $\hat{x}^{*\top} = [\hat{x}_1^{*\top}, \hat{x}_2^{*\top}, \hat{x}_3^*]$ are solutions to problems (47) and (48), respectively, are

$$u_1^* + A_{11}\hat{x}_1^* + A_{12}\hat{x}_2^* - b_1\hat{x}_3^* \geq 0_{m_1}, \quad D(u_1^*)(u_1^* + A_{11}\hat{x}_1^* + A_{12}\hat{x}_2^* - b_1\hat{x}_3^*) = 0, \quad u_1^* \geq 0_{m_1}, \tag{49}$$

$$u_2^* + A_{21}\hat{x}_1^* + A_{22}\hat{x}_2^* - b_2\hat{x}_3^* = 0_{m_2}, \tag{50}$$

$$A_{11}^\top u_1^* + A_{21}^\top u_2^* \leq 0_{n_1}, \quad D(\hat{x}_1^*)(A_{11}^\top u_1^* + A_{21}^\top u_2^*) = 0_{n_1}, \quad \hat{x}_1^* \geq 0_{n_1} \tag{51}$$

$$A_{12}^\top u_1^* + A_{22}^\top u_2^* = 0_{n_2}, \tag{52}$$

$$\rho - b_1^\top u_1^* - b_2^\top u_2^* = 0. \tag{53}$$

It follows from (49) and (50) that u^* and $\hat{x}^*$ are related by the equations

$$u_1^* = (b_1\hat{x}_3^* - A_{11}\hat{x}_1^* - A_{12}\hat{x}_2^*)_+, \qquad u_2^* = b_2\hat{x}_3^* - A_{21}\hat{x}_1^* - A_{22}\hat{x}_2^*.$$

Using them and equating the optimal values of the objective functions of the primal and dual problems (47) and (48), we find that $\|u^*\|^2 = \rho\hat{x}_3^*$. Since $U \neq \emptyset$ and $u^* \in U$, it holds that $\|u^*\| \neq 0$ by virtue of the condition $b^\top u^* = \rho > 0$. Hence, $\hat{x}_3^* > 0$.

Changing variables in (49) – (52),

$$u^* = \hat{x}_3^* z^*, \qquad \hat{x}_1^* = \hat{x}_3^* x_1^*, \qquad \hat{x}_2^* = \hat{x}_3^* x_2^*,$$

and cancelling out the common factor $\hat{x}_3^*$ in the expressions obtained, we arrive at the Kuhn–Tucker conditions (26) – (30) for problem (5) at the point $[z^*, x^*]$. Substituting $u^* = \hat{x}_3^* z^*$ into (53) and taking into account (16), we obtain

$$\rho/\hat{x}_3^* - b^\top z^* = \rho/\hat{x}_3^* - \|z^*\|^2.$$

Hence, it follows that $u^* = \hat{x}_3^* z^* = \rho z^*/\|z^*\|^2 = \tilde{u}^*$ for $\hat{x}_3^* = \rho/\|z^*\|^2$; i.e., the normal solution to system (II) is given by the former expression in (45).

The former expression in (45) implies that $\rho = \|\tilde{u}^*\| \, \|z^*\|$ and

$$z^* = \tilde{u}^* \|z^*\|^2/\rho = \tilde{u}^* \rho^2/(\rho\|\tilde{u}^*\|^2) = \rho\tilde{u}^*/\|\tilde{u}^*\|^2,$$

which proves the latter formula in (45).

If $U = \emptyset$, then $\operatorname{pen}(u^*, U) \neq 0$. The vector $w^* \in W$ determined from (33) satisfies the condition $\|w^*\| \neq 0$, and $w_3^* > 0$ by virtue of (34). It follows from (42) and (43) that the vector $x^* = [x_1^*, x_2^*]$ with the components

$$x_1^* = w_1^*/w_3^*, \qquad x_2^* = w_2^*/w_3^* \tag{54}$$

is a solution to system (I).

Let $\tilde{x}^*$ be the normal solution to system (I), i.e., a solution to the problem

$$\min_{x \in X} \|z\|^2. \tag{55}$$

The Lagrange function for this problem is

$$L(x, \mu) = \|x\|^2/2 + \mu_1^\top (b_1 - A_{11}x_1 - A_{12}x_2) + \mu_2^\top (b_2 - A_{21}x_1 - A_{22}x_2),$$

and the Kuhn–Tucker conditions calculated at the saddle point $[\tilde{x}^*, \mu^*]$, where $\tilde{x}^{*\top} = [\tilde{x}_1^{*\top}, \tilde{x}_2^{*\top}]$ is a solution to problem (55) and $\mu^{*\top} = [\mu_1^{*\top}, \mu_2^{*\top}]$ is the optimum vector of Lagrange multipliers, are

$$\tilde{x}_1^* - A_{11}^\top \mu_1^* - A_{21}^\top \mu_2^* \geq 0_{n_1}, \quad D(\tilde{x}_1^*)(\tilde{x}_1^* - A_{11}^\top \mu_1^* - A_{21}^\top \mu_2^*) = 0_{n_1}, \quad \tilde{x}_1^* \geq 0_{n_1},$$
$$\tilde{x}_2^* - A_{12}^\top \mu_1^* - A_{22}^\top \mu_2^* = 0_{n_2},$$
$$b_1 - A_{11}\tilde{x}_1^* - A_{12}\tilde{x}_2^* \leq 0_{m_1}, \quad D(\mu_1^*)(b_1 - A_{11}\tilde{x}_1^* - A_{12}\tilde{x}_2^*) = 0_{m_1}, \quad \mu_1^* \geq 0_{m_1},$$
$$b_2 - A_{21}\tilde{x}_1^* - A_{22}\tilde{x}_2^* = 0_{m_2}.$$

These conditions are obtained from the Kuhn–Tucker conditions (39), (40), (42), and (43) for problem (6) by dividing the latter by w_3^* and introducing $x_1^* = w_1^*/w_3^*$, $x_2^* = w_2^*/w_3^*$, $\mu_1^* = u_1^*/w_3^*$, and $\mu_2^* = u_2^*/w_3^*$. Hence, the vector $\tilde{x}^*$ whose components are defined by (54) is the normal solution to system (I). The theorem is proved. $\square$

Thus, Theorem 3 reduces the problem of solvability of system (I) or (II) to minimizing the residual of either system. If the norm of the minimum residual is nonzero, then the system is inconsistent and the residual can be used in simple formulas to find the normal solution to the consistent system.

If weighted residuals are used in problem (1) or (2), then the multipliers of these residuals must be taken into account in both alternative systems and formulas (45) and (46) can be used to calculate the normal solutions to these modified systems.

The alternative system (II) admits various representations. According to the theorems stated above, the system alternative to (I) is obtained from the adjoint system (I)$'$ by supplementing it with a condition that rules out the existence of the trivial solution to (5). For example, we may require that solutions of the adjoint system (I)$'$ satisfy the condition $b^\top u > 0$ (as in the Farkas lemma) or $b^\top u = 1$ (as in the Gale theorem), and so on. If system (I) does not contain inequalities, then it is sufficient to impose the restriction $\rho \neq 0$ on ρ in Theorem 3.

3. DETERMINATION OF THE STEEPEST DESCENT DIRECTION IN THE METHOD OF FEASIBLE DIRECTIONS

Below, we consider various systems that are special cases of (I) and (II). We refer to them as systems (I) and (II) indexed by the number of the corresponding section.

Let z^* be a nonzero solution to problem (5) and consider the normalized vectors $z_n = z/\|z^*\|$ and $z_n^* = z^*/\|z^*\|$. The feasible set of normalized vectors is defined as

$$Z_n = \{z_n \in \mathbb{R}^m : z_n \in Z, \ \|z_n\| = 1\},$$

where Z is the adjoint set in (I)$'$.

Consider the following auxiliary problem:

$$I_3 = \max_{z_n \in Z_n} b^\top z_n. \tag{56}$$

Theorem 4. *Let x^* be an arbitrary solution to problem* (1) *and z^* be the solution to problem* (5) *given by formula* (15)*. Let $\|z^*\| \neq 0$. Then, $z^*_n = z^*/\|z^*\|$ is a solution of problem* (56) *and*

$$I_3 = b^\top z^*_n = \|z^*\|. \tag{57}$$

Proof. The equality of the optimal values of the objective functions of the mutually dual problems (1) and (5) implies that

$$1/2 = \max_{z_n \in Z} \left[b^\top z_n/\|z^*\| - \|z_n\|^2/2 \right].$$

Since $Z_n \subset Z$, we have

$$\frac{1}{2} = \max_{z_n \in Z} \left[\frac{b^\top z_n}{\|z^*\|} - \frac{\|z_n\|^2}{2} \right] \geq \max_{z_n \in Z_n} \left[\frac{b^\top z_n}{\|z^*\|} - \frac{\|z_n\|^2}{2} \right] = \max_{z_n \in Z_n} \left[\frac{b^\top z_n}{\|z^*\|} - \frac{1}{2} \right].$$

Hence, it follows that

$$\|z^*\| \geq \max_{z_n \in Z_n} b^\top z_n. \tag{58}$$

Setting z_n equal to the vector $z^*/\|z^*\|$, which belongs to the feasible set Z_n, and using (16), we reduce (58) to an equation. The theorem is proved. $\square$

Problem (56) arises when the nonlinear programming problem

$$\min_{p \in P} f(p), \qquad P = \{p \in \mathbb{R}^m : h(p) \leq 0_{n_1}, \;\; g(p) = 0_{n_2}\} \tag{59}$$

is solved by the method of feasible directions. Here, $f : \mathbb{R}^m \to \mathbb{R}^1$, $h : \mathbb{R}^m \to \mathbb{R}^{n_1}$, $g : \mathbb{R}^m \to \mathbb{R}^{n_2}$; the functions $f(p)$, $h(p)$, and $g(p)$ are continuously differentiable; the set P is not empty; and problem (59) has a solution.

Let $p \in P$ be an arbitrary fixed feasible point. We introduce a vector $x \in \mathbb{R}^n$ of the Lagrange multipliers, $x^\top = [x_1^\top, x_2^\top]$, where $x_1 \in \mathbb{R}^{n_1}_+$, $x_2 \in \mathbb{R}^{n_2}$, and $n = n_1 + n_2$; define the Lagrange function as

$$L(p, x) = f(p) + h^\top(p)x_1 + g^\top(p)x_2$$

and introduce the complementary slackness conditions

$$x_1^i h^i(p) = 0, \qquad 1 \leq i \leq n_1. \tag{60}$$

A component $h^i(p)$ of the vector $h(p)$ is said to be active at the point $p \in P$ if $h^i(p) = 0$. By virtue of (60), all components of the vector x_1 corresponding to the inactive components of $h(p)$ vanish. For simplicity, we assume that all components of the vector $h(p)$ in the Lagrange function are active. The Kuhn–Tucker conditions for problem (59) at the point $[p, x]$, where $p \in P$, are

$$L_p(p, x) = f_p(p) + h_p(p)x_1 + g_p(p)x_2 = 0_m, \qquad x_1 \geq 0_{n_1}. \tag{I$_3$}$$

If $p \in P$ is fixed, these equations in x may be interpreted as a special case of system (I).

We introduce the vector $p' = p + \tau z$, where τ is a step along the descent direction $z \in \mathbb{R}^m$ ($\|z\| = 1$), and linearize the objective and constraint functions in problem (59). Assuming that

τ is small and neglecting the terms of higher order, we arrive at the following problem of finding the steepest descent direction: направления наискорейшего спуска:

$$I_4 = \min_{z \in \hat{Z}_n} z^\top f_p(p), \quad \hat{Z}_n = \{z \in \mathbb{R}^m : h_p^\top(p) z \leq 0_{n_1}, \ \ g_p^\top(p) z = 0_{n_2}, \ \ \|z\| = 1\}. \tag{61}$$

If problem (61) has a solution z_n^* such that $I_4 < 0$, then this direction is said to be the *steepest descent direction*. This implies that, at least in the linear approximation, the point p can be improved by taking a new vector p'. When the step τ is sufficiently small, the vector p' belongs to the feasible set P and $f(p') < f(p)$. If problem (61) has no such solution, then the point p cannot be improved locally.

To make use of the results obtained earlier, we assume that $h_p^\top(p) = A_{21}^\top$, $g_p^\top(p) = A_{22}^\top$, and $-f_p(p) = b_2$, while the remaining submatrices of A and vector b_1 are zero. Then, system (II) alternative to (I) can be written as

$$u^\top h_p(p) \leq 0_{n_1}^\top, \quad u^\top g_p(p) = 0_{n_2}^\top, \quad -u^\top f_p(p) = \rho > 0. \tag{II$_3$}$$

If system (I)$_3$ is solvable, then, by Theorem 3, its normal solution has the form $\tilde{u}^* = \rho z^*/\|z^*\|^2$, where $z^* = -L_p(p, x^*)$, and x^* is found by solving the unconstrained minimization problem (1), which has the following form in this given case:

$$I_1 = \min_{x_1 \in \mathbb{R}_+^{n_1}} \min_{x_2 \in \mathbb{R}^{n_2}} \|L_p(p, x)\|^2/2. \tag{62}$$

Normalizing the vector $\tilde{u}^*$, we obtain $\tilde{u}_n^* = z^*/\|z^*\| = z_n^*$. The vector z_n^* belongs to $\hat{Z}_n$, and $I_4 = -I_3 = -\|z^*\|$ by Theorem 4 (see Eq. (57)), which implies that z_n^* is the steepest descent direction for the linearized problem (61). This direction exists if and only if system (I)$_3$ with a fixed $p \in P$ cannot be solved for the Lagrange multipliers $x_1 \in \mathbb{R}_+^{n_1}$ and $x_2 \in \mathbb{R}^{n_2}$ of problem (59), i.e., if $I_1 > 0$.

Thus, to determine the steepest descent direction, it is not necessary to solve the constrained minimization problem (61). This direction is found by solving the unconstrained minimization problem (62).

Note that this approach is particularly efficient when the number n of active constraints at $p \in P$ is considerably less than the dimension of p, because the minimization problem (62) is solved in the n-dimensional space. At the point p', the set of active constraints is updated (and denoted by $h(p)$ again).

4. PROJECTION AND CORRECTION PROBLEMS

Let us give a geometric interpretation of the results obtained. By (16), the residual vector z^* is orthogonal to the vector $b - z^*$. Hence, the origin in $\mathbb{R}^m$ and the points z^* and b make up a rectangular triangle in which b is the hypotenuse. The vectors z^* and $b - z^*$ are the legs of length $\mathrm{pen}\,(x^*, X)$ and $\mathrm{dist}\,(b, Z)$, respectively. Then, relation (19) follows from the Pythagorean theorem. Let $b^\perp$ be the projection of b onto the set Z. Then, the vector $b^\| = b - b^\perp$ orthogonal to it is the sum of two vectors, $b_1^\| = b_1 - (b_1 - A_{11}x_1^* - A_{12}x_2^*)_+$ and $b_2^\| = A_{21}x_1^* + A_{22}x_2^*$. It follows from (16) and (17) that, for $b_1^\perp \perp b_1^\|$ and $b_2^\perp \perp b_2^\|$, Eqs. (18) have the form

$$z^* = b^\perp = \mathrm{pr}\,(b, Z), \qquad \|b^\perp\| = \mathrm{pen}\,(x^*, X), \qquad \|b^\|\| = \mathrm{dist}\,(b, Z). \tag{63}$$

In this notation, Eqs. (16) and (19) obviously become $\|b^\perp\|^2 = \|b^\perp\|^2$ and $\|b^\perp\|^2 + \|b^\|\|^2 = \|b\|^2$, respectively. If Z is a linear subspace, then $b^\perp$ is the projection of b onto Z and $b^\|$ is the projection of b onto the orthogonal complement to Z.

Similarly, in Theorem 3,

$$w^* = r^\perp = \mathrm{pr}\,(r, W), \quad \|r^\perp\| = \mathrm{pen}\,(u^*, U), \quad r^\| = r - r^\perp, \quad \|r^\|\| = \mathrm{dist}\,(r, W). \tag{64}$$

It follows from Theorems 2 and 3 and from relations (63) and (64) that conditions (3) in the criterion for alternativity can be represented as

$$\|b^\perp\|\|r^\perp\| = \|z^*\|\|w^*\| = 0, \qquad \|b^\perp\| + \|r^\perp\| = \|z^*\| + \|w^*\| > 0.$$

Hence, systems (I) and (II) are mutually alternative.

It follows from Theorem 1 that $\|z^*\| \leq \|b\|$, $b^\top z^* \geq 0$. The vector z^* belongs to the hemisphere of radius $\|b\|$ centered at the origin in $\mathbb{R}^m$ where the vectors z^* and b make an acute angle. It is evident that the vectors $r^\perp$ and $r^\|$ belong to the sphere of radius ρ centered at the origin in $\mathbb{R}^{n+1}$.

Consider problem (14), i.e., the problem of finding the projection of a point $\bar{x} \in \mathbb{R}^n$ onto a nonempty set X. By Theorem 3 the normal solution to system (I) is the projection of the origin onto the set X: $\tilde{x}^* = \mathrm{pr}\,(0_n, X)$. Changing variables in (14), $y = x - \bar{x}$, we reduce it to the problem of projecting the origin onto the "shifted" set

$$\bar{X} = \{y \in \mathbb{R}^n : A_{11}y_1 + A_{12}y_2 \geq \bar{b}_1, \;\; A_{21}y_1 + A_{22}y_2 = \bar{b}_2, \;\; y_1 \geq -\bar{x}_1\}, \tag{65}$$

where $\bar{b}_1 = b_1 - A_{11}\bar{x}_1 - A_{12}\bar{x}_2$ and $\bar{b}_2 = b_2 - A_{21}\bar{x}_1 - A_{22}\bar{x}_2$.

Then, problem (14) takes the form

$$J = \min_{y \in \bar{X}} \|y\| = \|y^*\| = \|\,\mathrm{pr}\,(0_n, \bar{X})\|. \tag{66}$$

The solutions to problems (14) and (66) are related by the simple equation

$$\bar{x}^* = \mathrm{pr}\,(\bar{x}, X) = \bar{x} + y^*. \tag{67}$$

Similarly, it can be shown that, if $U \neq \emptyset$, then the normal solution to system (II) is $\tilde{u}^* = \mathrm{pr}\,(0_m, U)$, and

$$\bar{u}^* = \mathrm{pr}\,(\bar{u}, U) = \bar{u} + v^*, \tag{68}$$

where $v^* = \mathrm{pr}\,(0_n, \bar{U})$, $\bar{U} = \{v \in \mathbb{R}^m : A_{11}^\top v_1 + A_{21}^\top v_2 \leq d_1, \;\; A_{12}^\top v_1 + A_{22}^\top v_2 = d_2, \;\; b_1^\top v_1 + b_2^\top v_2 = d_3, \;\; v_1 \geq \bar{u}_1\}$ and $d_1 = -A_{11}^\top \bar{u}_1 - A_{21}^\top \bar{u}_2$, $d_2 = -A_{12}^\top \bar{u}_1 - A_{22}^\top \bar{u}_2$, and $d_3 = \rho - b_1^\top \bar{u}_1 - b_2^\top \bar{u}_2$.

Problems of optimal correction of linear inconsistent systems were stated in [11, 12]. With regard to system (I), the problem is to find a vector $\tilde{b}$ with a minimum Euclidean norm such that the substitution of the vector $b - \tilde{b}$ for b makes the inconsistent system (I) a consistent one.

Theorem 5. *Let x^* be on arbitrary solution to problem* (1) *and z^* be the minimum residual vector calculated at the point x^*. Then, the optimal correction of system* (I) *consists in the replacement of the vector b by $b - z^*$. The pseudosolution x^* to system* (I) *is a solution to the corrected system*

$$A_{11}x_1 + A_{12}x_2 \geq b_1 - z_1^*, \quad A_{21}x_1 + A_{22}x_2 = b_2 - z_2^*, \quad x_1 \geq 0_{n_1}. \tag{69}$$

By virtue of (63), the components of z^* can be represented as $z_1^* = b_1^\perp$, $z_2^* = b_2^\perp$. Then $b_1^\| = b_1 - b_1^\perp$, $b_1^\| \leq b_1$, $b_2^\| = b_2 - b_2^\perp$, and (69) takes the form

$$A_{11}x_1 + A_{12}x_2 \geq b_1^\|, \quad A_{21}x_1 + A_{22}x_2 = b_2^\|, \quad x_1 \geq 0_{n_1}. \tag{70}$$

After substituting $b^{\|}$ for b, the alternative system (II) becomes inconsistent. By Theorem 3, minimizing its residual, we find the normal solution $\tilde{x}^*$ to the corrected system (70). Thus, an inconsistent system (I) is corrected and the normal solution to the corrected system (70) is found by solving two unconstrained minimization problems in $\mathbb{R}^n$ and $\mathbb{R}^m$. The m-dimensional vector $\tilde{b} = z^*$ with a minimal Euclidean norm is found by solving the unconstrained minimization problem (1) in n variables, and the n-dimensional normal vector $\tilde{x}^*$ that solves the corrected system (70) is found by solving the unconstrained minimization problem (1) in n variables.

In another method for correcting an inconsistent system (I), a single unconstrained minimization problem in $\mathbb{R}^m$ is solved instead of the two aforementioned unconstrained minimization problems. We represent the problem of correcting system (I) and finding its solution as that of finding the normal solution to the consistent system

$$A_{11}x_1 + A_{12}x_2 + \tilde{b}_1 \geq b_1, \quad A_{21}x_1 + A_{22}x_2 + \tilde{b}_2 = b_2, \quad x_1 \geq 0_{n_1}, \quad \tilde{b}_1 \geq 0_{m_1}, \tag{71}$$

which is obtained from (I) by introducing additional variables $\tilde{b}_1 \in \mathbb{R}^{m_1}_+$ and $\tilde{b}_2 \in \mathbb{R}^{m_2}$. This system is always consistent. Therefore, the alternative system

$$\begin{gathered} A_{11}^\top u_1 + A_{21}^\top u_2 \leq 0_{n_1}, \quad A_{12}^\top u_1 + A_{22}^\top u_2 = 0_{n_2}, \quad u_1 \leq 0_{m_1}, \quad u_2 = 0_{m_2}, \\ b_1^\top u_1 + b_2^\top u_2 = \rho > 0, \quad u_1 \geq 0_{m_1} \end{gathered}$$

is always inconsistent.

Having solved the problem of residual minimization for the alternative system,

$$\min_{u_1 \in \mathbb{R}^{m_1}_+} \min_{u_2 \in \mathbb{R}^{m_2}} \frac{\|(A_{11}^\top u_1 + A_{21}^\top u_2)_+\|^2 + \|A_{12}^\top u_1 + A_{22}^\top u_2\|^2 + \|u_1\|^2 + \|u_2\|^2 + (\rho - b_1^\top u_1 - b_2^\top u_2)^2}{2},$$

we use formula (46) to obtain the normal solution of system (71)

$$\begin{aligned} \tilde{x}_1^* &= \frac{(A_{11}^\top u_1^* + A_{21}^\top u_2^*)_+}{w_3^*}, \qquad \tilde{x}_2^* = \frac{A_{12}^\top u_1^* + A_{22}^\top u_2^*}{w_3^*}, \\ \tilde{b}_1^* &= u_1^*/w_3^*, \quad \tilde{b}_2^* = u_2^*/w_3^*, \qquad w_3^* = \rho - b_1^\top u_1^* - b_2^\top u_2^*. \end{aligned}$$

5. SYSTEMS OF LINEAR EQUATIONS

Consider the special case when systems (I) and (II) do not contain inequalities. Then, system (I), which determines the set X, has the form

$$Ax = b, \tag*{(I)$_5$}$$

and the alternative system, which determines the set U, is written as

$$A^\top u = 0_n, \qquad b^\top u = \rho \neq 0.$$

For the sake of convenience, the latter system is represented as

$$\hat{A}^\top u = r, \tag*{(II)$_5$}$$

where $\hat{A} = [-A, b]$ and $r^\top = [0_n^\top, \rho]$, $\rho \neq 0$.

By Theorem 3, we have the Fredholm alternative: only one of these systems—either (I)$_5$ or (II)$_5$ — has a solution.

We represent b as the sum of two orthogonal vectors, $b = b^{\|} + b^{\perp}$, where $b^{\|} = \mathrm{pr}\,(b, \mathrm{im}\ A)$, $b^{\perp} = (b, \ker A^{\top})$, and $Z = \ker A^{\top}$. If $\|b^{\perp}\| = 0$, then $X \neq \emptyset$ and $U = \emptyset$. If $\|b^{\perp}\| \neq 0$, then $X = \emptyset$ and $U \neq \emptyset$.

To find out which system is solvable and to solve it, it is sufficient to find either x^* or u^* by solving one of the following unconstrained minimization problems:

$$\min_{x \in \mathbb{R}^n} \frac{\|b - Ax\|^2}{2} = \frac{\|b - Ax^*\|^2}{2}, \qquad \min_{u \in \mathbb{R}^m} \frac{\|r - \hat{A}^{\top} u\|^2}{2} = \frac{\|r - \hat{A}^{\top} u^*\|^2}{2}. \tag{72}$$

These problems can be interpreted as the application of the least-squares method to systems $(\mathrm{I})_5$ and $(\mathrm{II})_5$. By Theorems 1 – 3, we have $z^* = b - Ax^*$, $w^{*\top} = = [w_2^{*\top}, w_3^*] = [r - \hat{A}^{\top} u^*\|^{\top}$, $w_2^* = A^{\top} u^*$, and $w_3^* = \rho - b^{\top} u^*$. The necessary and sufficient conditions for the existence of minima in problems (72), the so-called "normal equations", are

$$A^{\top}(b - Ax^*) = 0_n, \qquad \hat{A}(r - \hat{A}^{\top} u^*) = 0_m. \tag{73}$$

From these equations, we determine the residual vectors $z^* = b^{\perp} \in \ker A^{\top}$ and $w^* = r^{\perp} \in \ker \hat{A}$. The vectors x^* and u^* satisfying (73) are pseudosolutions to systems $(\mathrm{I})_5$ and $(\mathrm{II})_5$, respectively.

If $\|b^{\perp}\| = 0$, then $X \neq \emptyset$, $U = \emptyset$, $b = b^{\|}$, $z^* = 0_m$, $w_3^* \neq 0$, and the normal solution is

$$\tilde{x}^* = A^{\top} u^* / (\rho - b^{\top} u^*), \tag{74}$$

where u^* is a pseudosolution to system $(\mathrm{II})_5$. Substituting it into $(\mathrm{I})_5$ and performing simple calculations, we obtain

$$(AA^{\top} + bb^{\top}) u^* = \rho b. \tag{75}$$

If $\|b^{\perp}\| \neq 0$, then $X = \emptyset$, $U \neq \emptyset$, and the normal solution of system $(\mathrm{II})_5$ is

$$\tilde{u}^* = \rho(b - Ax^*)/\|b - Ax^*\|^2, \tag{76}$$

where x^* is a pseudosolution to system $(\mathrm{I})_5$. System $(\mathrm{I})_5$ corrected by using the second formula in (69) can be represented as $Ax = Ax^* = b^{\|}$. Its normal solution x^* is given by (74), where $b = b^{\|}$ and u^* is found by solving the second problem in (72) with $b = b^{\|}$. If the rank of the matrix A is n, then the corrected system has a unique solution x^* equal to its normal solution $\tilde{x}^*$.

In what follows, we assume that the $m \times n$ matrix A has the maximum possible rank. Denote by A^+ the $n \times m$ pseudoinverse of A. Consider two special cases in which problems (1), (2), (5), and (6) can be solved analytically.

Case 1: Let $\mathrm{rank}\ A = m$.

Then, $n \geq m$, $X \neq \emptyset$, $\|b^{\perp}\| = 0$ and $U = \emptyset$, $A^+ = A^{\top}(AA^{\top})^{-1}$, $AA^+ = I_m$, the rows of A are linearly independent, $(A^{\top})^{\|} = A^+ A$ is the $n \times n$ matrix of projection onto $\mathrm{im}\ A^{\top}$, and $I_n - (A^{\top})^{\|} = (A^{\top})^{\perp}$ is the matrix of projection onto $\ker A$. By the Kronecker–Capelli theorem, $\mathrm{rank}\ A = \mathrm{rank}\ \hat{A} < \mathrm{rank}\,[\hat{A}^{\top}, r] = m + 1$. The normal solution $\tilde{x}^*$ to system $(\mathrm{I})_5$ can be represented in several forms:

$$\tilde{x}^* = A^+ b = A^{\top}(AA^{\top})^{-1} b = \mathrm{pr}\,(0_n, X) = (A^{\top})^{\|} x, \tag{77}$$

where x is an arbitrary vector in X.

The number of rows in the matrix$\hat{A}^{\top}$ is greater than the number of columns. Therefore, $(\hat{A}^{\top})^+ = (\hat{A}\hat{A}^{\top})^{-1}\hat{A}$, $(\hat{A}^{\top})^+ \hat{A}^{\top} = I_m$, $(\hat{A}^{\top})^{\|} = \hat{A}^{\top}(\hat{A}^{\top})^+$ is the square matrix of order $n + 1$ of projection onto im $\hat{A}^\top$, and $(\hat{A}^\top)^\perp = I_{n+1} - (\hat{A}^\top)^\|$. A pseudosolution to system $(\mathrm{II})_5$ has the form

$$u^* = (\hat{A}^\top)^+ r = (\hat{A}\hat{A}^\top)^{-1}\hat{A}r. \tag{78}$$

The vector u^* satisfies the corrected system $\hat{A}^\top u^* = r^\|$, where $r^\|$ is the projection of the vector r onto the subspace im $\hat{A}^\top$,

$$r^\| = \mathrm{pr}\,(r, \mathrm{im}\ \hat{A}^\top) = (\hat{A}^\top)^\| r = \hat{A}^\top(\hat{A}\hat{A}^\top)^{-1}\hat{A}r.$$

The vector w^* is the projection of r onto the orthogonal complement $\ker \hat{A}$ to the subspace im $\hat{A}^\top$. Indeed, performing some straightforward transformations, one can show that

$$w^* = (\hat{A}^\top)^\perp r = r^\perp.$$

Let us introduce the square nonsingular matrix $\Phi = (\hat{A}\hat{A}^\top)^{-1} = (AA^\top + +bb^\top)^{-1}$ of order m. Then, it follows from (78) that

$$u^* = \rho\Phi b. \tag{79}$$

This formula can also be derived from (75) by assuming that the matrix on the left-hand side of (75) is invertible. Substituting (79) into (74), we obtain

$$\tilde{x}^* = A^\top\Phi b/(1 - b^\top\Phi b). \tag{80}$$

Since the normal solution is unique, we can equate (77) to (80). As a result, we obtain the matrix identity

$$A^\top\Phi b/(1 - b^\top\Phi b) = A^\top(AA^\top)^{-1}b. \tag{81}$$

Using the optimal vector of Lagrange multipliers $\mu^* \in \mathbb{R}^m$ for the problem of finding the normal solution to the consistent system $(\mathrm{I})_5$, we represent (77) as

$$\tilde{x}^* = A^\top\mu^*, \qquad \mu^* = (AA^\top)^{-1}b. \tag{82}$$

By virtue of (80), we have

$$\mu^* = \Phi b/(1 - b^\top u^*). \tag{83}$$

From (82) and (83), we obtain the identity

$$\Phi b = (1 - b^\top\Phi b)(AA^\top)^{-1}b. \tag{84}$$

Let γ be the least eigenvalue of the matrix $AA^\top$. Since rank $A = m$, we have $\gamma > 0$ and (82) implies that

$$\begin{aligned} \gamma\|\mu^*\|^2 &\le \|\tilde{x}^*\|^2 = \mu^{*\top}AA^\top\mu^* = \mu^{*\top}b \le \|b\|\|\mu^*\|, \\ \|\mu^*\| &\le \|b\|/\gamma, \quad \|\tilde{x}^*\| \le \|b\|/\sqrt{\gamma}. \end{aligned} \tag{85}$$

Let $\bar{x}^*$ be the projection of $\bar{x}$ onto X. Then, $\bar{x}^* - \bar{x} = \mathrm{pr}\,(0_n, \bar{X})$ by virtue of (67). According to (65), the set $\bar{X}$ can be represented as $\bar{X} = \{y \in \mathbb{R}^n : Ay = \bar{b}\}$, where $\bar{b} = b - A\bar{x}$. Since $\mathrm{pr}\,(0_n, \bar{X})$ is a vector from $\bar{X}$ with the least Euclidean norm, we obtain (cf. (77))

$$\bar{x}^* - \bar{x} = A^\top(AA^\top)^{-1}\bar{b}. \tag{86}$$

By analogy with (82) and (85), we have

$$\begin{gathered} \bar{x}^* = A^+\bar{b} = A^\top\bar{\mu}^*, \quad \bar{\mu}^* = (AA^\top)^{-1}\bar{b}, \\ \gamma\|\bar{\mu}^*\|^2 \le \|\bar{x}^*\|^2 = \bar{\mu}^{*\top}AA^\top\bar{\mu}^* = \bar{\mu}^{*\top}\bar{b} \le \|\bar{\mu}^*\|\|\bar{b}\|, \quad \|\bar{\mu}^*\| \le \|\bar{b}\|/\gamma, \\ \mathrm{dist}\,(\bar{x}, X) = \|\bar{x}^* - \bar{x}\| = (\bar{\mu}^\top\bar{b})^{1/2} \le \|\bar{b}\|/\sqrt{\gamma} = \mathrm{pen}\,(\bar{x}, X)/\sqrt{\gamma}. \end{gathered} \tag{87}$$

Relations (86) and (87) have particularly simple forms when $m = 1$. In this case, $b \in \mathbb{R}^1$, and A is an n-dimensional row vector $a^\top$. Formula (86) specifies the projection of $\bar{x}$ onto the plane $a^\top x = b$, and (87) determines the distance from $\bar{x}$ to this plane; i.e.,

$$\bar{x}^* = \bar{x} + a(b - a^\top \bar{x})/\|a\|^2, \quad \text{dist}\,(\bar{x}, X) = |\,b - a^\top \bar{x}|/\|a\|. \tag{88}$$

If $\bar{x} = 0_n$, then $\bar{x}^* = ab/\|a\|^2$, and $\|\tilde{x}^*\| = |\,b|/\|a\|$.

Case 2: rank $A = n$.

In this case, $n \le m$. Let $U \neq \emptyset$ and $\|b^\perp\| = 0$. Then, $X = \emptyset$. Since the columns of the matrix A are linearly independent, we have $A^+ = (A^\top A)^{-1}A^\top$, $A^+A = I_n$, $A^\| = AA^+$ is the $m \times m$ matrix of the projection onto im A, and $I_m - A^\| = A^\perp$ is the matrix of the projection onto the orthogonal subspace ker $A^\top$. By the Kronecker–Capelli theorem, rank $\hat{A} = n + 1$. A pseudosolution to system $(\mathrm{I})_5$ and the normal solution to $(\mathrm{II})_5$ are written as

$$x^* = A^+b, \quad A^+ = (A^\top A)^{-1}A^\top, \quad \tilde{u}^* = \hat{A}(\hat{A}^\top \hat{A})^{-1}r = \text{pr}\,(0_m, U) = (\hat{A})^\| u, \tag{89}$$

where $u \in U$. The vector x^* satisfies the corrected consistent system $Ax^* = b^\|$, where $b^\| = \text{pr}\,(b, \text{im } A) = A(A^\top A)^{-1}A^\top b$ is the projection of b onto the subspace im A. The vector z^* is the projection of b onto ker $A^\top$ (the orthogonal complement to im A). Indeed, it can be shown that $z^* = A^\perp b = b^\perp$.

The square matrix $\hat{A}^\top \hat{A}$ of order $n+1$ in (89) is nonsingular and can be represented in the block form

$$\hat{A}^\top \hat{A} = \begin{bmatrix} A^\top A & -A^\top b \\ -b^\top A & b^\top b \end{bmatrix}.$$

Its inverse is determined by applying the Frobenius formula, and (89) yields

$$\begin{gathered} (\hat{A}^\top \hat{A})^{-1} r = \rho\beta \begin{bmatrix} -HA^\top b \\ 1 + \beta b^\top AHA^\top b \end{bmatrix}, \\ \tilde{u}^* = \beta\rho\left[(1 + \beta b^\top AHA^\top b)b - AHA^\top b\right], \end{gathered} \tag{90}$$

where $\beta = 1/\|b\|^2$ and $H = (A^\top A - \beta A^\top bb^\top A)^{-1}$ is a square matrix of order n. Equating expressions (76) and (90) for $\tilde{u}^*$, we obtain the second matrix identity

$$\frac{\left[I_m - A(A^\top A)^{-1}A^\top\right] b}{\|\left[I_m - A(A^\top A)^{-1}A^\top\right] b\|^2} = \beta\left[(1 + \beta b^\top AHA^\top b)b - AHA^\top b\right]. \tag{91}$$

The matrix identities (81), (84), and (91) can be proved without invoking Theorems 1 – 3.

Denote by η the least eigenvalue of the matrix $\hat{A}^\top \hat{A}$. It follows from (89) that

$$\begin{gathered} \tilde{u}^* = \hat{A}\xi^*, \quad \xi^* = (\hat{A}^\top \hat{A})^{-1}r, \\ \eta\|\xi^*\|^2 \le \|\tilde{u}^*\|^2 = \xi^{*\top}\hat{A}^\top \hat{A}\xi^* \le \xi^{*\top} r \le \|\xi^*\|/\rho, \\ \|\xi^*\| < 1/(\rho\eta), \quad \|\tilde{u}^*\| \le 1/(\rho\sqrt{\eta}), \\ \text{dist}\,(\bar{u}, U) = \|\bar{u}^* - \bar{u}\| \le \|r - \hat{A}^\top \bar{u}\|/\sqrt{\eta}. \end{gathered} \tag{92}$$

Using (68) and the last formula in (89), we obtain an expression for the projection of $\bar{u}$ onto the nonempty set U:

$$\bar{u}^* = \text{pr}\,(\bar{u}, U) = \bar{u} + \text{pr}\,(0_m, \bar{U}) = \bar{u} + \hat{A}(\hat{A}^\top \hat{A})^{-1}r.$$

Let the system determining the set X have the form

$$Ax \geq b, \tag{I$_6$}$$

where A is an $m \times n$ matrix, $b \in \mathbb{R}^m$, and $\|b\| \neq 0$. The alternative system determining the set U is written as

$$\hat{A}^\top u = r, \quad u \geq 0_m, \tag{II$_6$}$$

where (as in Section 5) the augmented matrix has the form $\hat{A} = [-A, b]$ and $r^\top = [0_m^\top, \rho]$, where $\rho > 0$.

According to Theorem 3, only one of systems (I)$_6$ and (II)$_6$ is consistent. In the case of $\rho = 1$, this assertion is known as the Gale theorem.

The sets introduced above take the form

$$\begin{aligned} &Z = \{z \in \mathbb{R}^m_+, \;\; A^\top z = 0_n\}, && W = \{w \in \mathbb{R}^{n+1} : \hat{A}w \leq 0_m\}, \\ &\bar{X} = \{y \in \mathbb{R}^n : Ay = \bar{b} = b - A\bar{x}\}, && \bar{U} = \{v \in \mathbb{R}^m : \hat{A}^\top v = \bar{r} = r - \hat{A}^\top \bar{u}\}, \\ &\mathrm{pen}\,(x, X) = \|(b - Ax)_+\|, && \mathrm{pen}\,(u, U) = \|r - \hat{A}^\top u\|. \end{aligned}$$

The vectors x^* and u^* are determined by solving the problems

$$\min_{x \in \mathbb{R}^n} \frac{\|(b - Ax)_+\|^2}{2} = \frac{\|(b - Ax^*)_+\|^2}{2}, \qquad \min_{u \in \mathbb{R}^m_+} \frac{\|r - \hat{A}^\top u\|^2}{2} = \frac{\|r - \hat{A}^\top u^*\|^2}{2}. \tag{93}$$

The necessary and sufficient optimality conditions for problems (93) are

$$A^\top (b - Ax^*)_+ = 0_n, \quad \hat{A}(r - \hat{A}^\top u^*) \leq 0_m, \quad D(u^*)\left[\hat{A}(r - \hat{A}^\top u^*)\right] = 0_m, \quad u^* \geq 0_m.$$

Hence, it follows that $z^* \in Z$ and $w^* \in W$.

By Theorems 1 – 3, we have $z^* = (b - Ax^*)_+$, $w^{*\top} = [w_2^{*\top}, w_3^*] = [r - \hat{A}^\top u^*]^\top$, $w_2^* = A^\top u^*$, and $w_3^* = \rho - b^\top u^*$. If $X \neq \emptyset$, then $w_3^* > 0$ and the normal solution of system (I)$_6$ is expressed as

$$\tilde{x}^* = A^\top u^* / (\rho - b^\top u^*). \tag{94}$$

This formula was derived in [8] for $\rho = 1$. Since $\tilde{x}^* \in X$, expression (94) leads to an analogue of formula (75): $(AA^\top + bb^\top)u^* \geq \rho b$.

If $X = \emptyset$, then $z^* > 0_m$ and the normal solution of system (II)$_6$ can be represented as

$$\tilde{u}^* = \rho(b - Ax^*)_+ / \|(b - Ax^*)_+\|^2 = \rho b^\perp / \|b^\perp\|^2. \tag{95}$$

In what follows, we assume that the $m \times n$ matrix A has the maximum possible rank. Denote by A^+ the $n \times m$ pseudoinverse matrix of A.

Case 1. Let $X \neq \emptyset$. Then $U = \emptyset$. Solving problem (2) (the second problem in (93)), we find the normal vector $\tilde{x}^* \in X$ from (94); hence, $A\tilde{x}^* \geq b$. Suppose that the first s conditions in this system of inequalities at the point $\tilde{x}^*$ are reduced to equations, and the remaining $c = m - s$ conditions are strict inequalities. Accordingly, we represent the matrix A and vectors b and u in the following partitioned form: $A^\top = [A_s^\top, A_c^\top]$, $b^\top = [b_s^\top, b_c^\top]$, $u^\top = [u_s^\top, u_c^\top]$. Consider the system

$$A_s x = b_s. \tag{96}$$

Let $s \le n$ and the rank of A_s be equal to s. Then, system (96) is solvable. By virtue of (77), its normal solution can be represented as

$$\tilde{x} = A_s^+ b_s = \mathrm{pr}\,(0_n, X_s),$$

where $A_s^+ = A_s^\top (A_s A_s^\top)^{-1}$ and $X_s = \{x \in \mathbb{R}^n : A_s x = b_s\}$. Since $\tilde{x}^* = \tilde{x}$, we have

$$A^\top u^*/(\rho - b^\top u^*) = A_s^\top u_s^*/(\rho - b_s^\top u_s^*) = A_s^\top (A_s A_s^\top)^{-1} b_s.$$

If all constraints at the point $\tilde{x}^*$ are active, we obtain the formulas derived in the preceding section. If $b \le 0_m$, then $\tilde{x}^* = 0_n$. When $b < 0_m$, there are no active constraints at the point $\tilde{x}^*$. Moreover, if rank $A = m$, then $u^* = 0_m$.

Consider the problem of finding the normal solution $\tilde{x}^* = \mathrm{pr}\,(0_n, X)$. The Lagrange function for this problem is $L(x, \mu) = \|x\|^2/2 + \mu^\top (b - Ax)$, and the Kuhn–Tucker conditions are

$$\tilde{x}^* = A^\top \mu^*, \quad A\tilde{x}^* \ge b, \quad \mu^{*\top}(A\tilde{x}^* - b) = 0, \quad \mu^* \ge 0_m.$$

Let us represent the optimal vector of Lagrange multipliers as $\mu^{*\top} = [\mu_s^{*\top}, \mu_c^{*\top}]$, where $\mu_s^* = (A_s A_s^\top)^{-1} b_s$ and $\mu_c^* = 0_c$. For simplicity, we assume that the first s constraints are active; i.e., $A_s \tilde{x}^* = \bar{b}_s$ and $A_c \tilde{x}^* < \bar{b}_c$. Then, $\mu^{*\top} A A^\top \mu^* = \mu_s^{*\top} A_s A_s^\top \mu_s^*$. Let $\bar{\gamma}$ be the least eigenvalue of the matrix $A_s A_s^\top$. It is evident that $\gamma \le \bar{\gamma}$, where γ is the least eigenvalue of $AA^\top$. In view of the condition rank $A = m$, it holds that $\bar{\gamma} \ge \gamma > 0$ and

$$\bar{\gamma}\|\mu_s^*\|^2 \le \mu_s^{*\top} A_s A_s^\top \mu_s^* = \mu_s^{*\top} b_s \ge 0, \qquad \|\mu_s^*\| \le \|b\|/\bar{\gamma},$$
$$\|\tilde{x}^*\|^2 = \|A_s^\top \mu_s^*\|^2 = \mu_s^{*\top} b_s \le \|b_s\|^2/\bar{\gamma}.$$

To find the projection of $\bar{x}$ onto the set X, we make use of the results of Section 4. Since $\mu^* = (AA^\top)^{-1}\bar{b}$ and $\bar{b} = b - A\bar{x}$, we obtain

$$\begin{aligned} [\mathrm{dist}\,(\bar{x}, X)]^2 &= \|\bar{x}^* - \bar{x}\|^2 = [\mathrm{pr}\,(\bar{x}, X)]^2 = [\mathrm{pr}\,(0_n, \bar{X})]^2 = \mu^{*\top}(b - A\bar{x}) \le \\ &\le \|\mu_s^*\| \|(b_s - A_s\bar{x})_+\| \le \|(b_s - A_s\bar{x})_+\|/\bar{\gamma} \le \|(b - A\bar{x})_+\|/\bar{\gamma}. \end{aligned} \tag{97}$$

Inequalities (87), (92), and (97) are analogous to Hoffman's inequalities [2, 3]. Note, however, that the inequalities obtained in this paper are different in some important respects. First, the coefficients γ, η, and $\bar{\gamma}$ are specified. Second, the distances between $\bar{x}^*$ and $\bar{x}$ and between $\bar{u}^*$ and $\bar{u}$ can be calculated by exact formulas. The fact that γ and $\bar{\gamma}$ do not depend on b and η is independent of r (see [3, Theorem 10.1] and [2, Lemma 35.5]) follows immediately from the analysis above.

Case 2. Let $X = \emptyset$. Then, $\|z^*\| \ne 0$, system $(\mathrm{I})_6$ is inconsistent, and some components of the vector $Ax^* - b$ are negative. The corrected system has the form

$$Ax \ge b^{\|} = b - (b - Ax^*)_+.$$

Hence, if $(b - Ax^*)^i \le 0$, then $(b^{\|})^i = b^i$; i.e., the ith component of b remains unchanged. If $(b - Ax^*)^i > 0$, then $(b^{\|})^i = (Ax^*)^i$ and b^i is replaced by $(Ax^*)^i$, which ensures the feasibility of the vector x^* in the corrected problem.

7. THE PROBLEM OF SEPARATING HYPERPLANES

Let us represent A, b, u, and z in the form

$$A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}, \qquad b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \qquad u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \qquad z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix},$$

where A_1 and A_2 are $k \times n$ and $\ell \times n$ matrices, respectively; $b_1, u_1, z_1 \in \mathbb{R}^k$; $b_2, u_2, z_2 \in \mathbb{R}^\ell$; and $k + \ell = m$. Assuming that the set X consists of the nonempty sets

$$X_1 = \{x \in \mathbb{R}^n : A_1 x \geq b_1\}, \qquad X_2 = \{x \in \mathbb{R}^n : A_2 x \geq b_2\}$$

such that $X_1 \cap X_2 = \emptyset$, we consider the problem of finding a hyperplane that strictly separates X_1 and X_2.

Let $\alpha \in [0, 1]$ be a scalar parameter.

Theorem 6 (on parallel separating hyperplanes). *Suppose that X_1 and X_2 are nonempty polyhedra, $X = X_1 \cap X_2 = \emptyset$, x^* solves the first problem in (93), and the components of the minimum residual vector $z^* = (b - Ax^*)_+$ are $z_1^* = (b_1 - A_1 x^*)_+$ and $z_2^* = (b_2 - A_2 x^*)_+$. Then, the following assertions are true:*

(i) *the parallel hyperplanes separating the sets X_1 and X_2 can be described by the two equivalent equations*

$$z_1^{*\top}(A_1 x - b_1) + \alpha \|z^*\|^2 = 0, \tag{98}$$
$$z_2^{*\top}(b_2 - A_2 x) + (\alpha - 1)\|z^*\|^2 = 0, \tag{99}$$

when $0 < \alpha < 1$, these hyperplanes strictly separate X_1 and X_2;

(ii) *if α is equal to*

$$\alpha^* = \|z_1^*\|^2 / \|z^*\|^2, \tag{100}$$

then x^* belongs to the separating hyperplane corresponding to this value of α;

(iii) *the distance d between the separating hyperplanes corresponding to $\alpha = 0$ and $\alpha = 1$ is*

$$d = \|z^*\|^2 / \|A_1^\top z_1^*\|.$$

Proof. By Theorem 1,

$$A^\top z = 0_n, \qquad b^\top z^* = \|z^*\|^2.$$

Premultiplying the first equation by an arbitrary vector $x \in \mathbb{R}^n$ and subtracting the second equation from the result, we obtain

$$z^{*\top}(Ax - b) + \|z^*\|^2 = 0. \tag{101}$$

We define a linear function $\varphi(x, \alpha)$ of $x \in \mathbb{R}^n$ depending on the parameter α by the following equivalent formulas:

$$\varphi(x, \alpha) = z_1^{*\top}(A_1 x - b_1) + \alpha \|z^*\|^2, \tag{102}$$
$$\varphi(x, \alpha) = z_2^{*\top}(b_2 - A_2 x) + (\alpha - 1)\|z^*\|^2. \tag{103}$$

For any fixed $\alpha \in [0, 1]$, the equation $\varphi(x, \alpha) = 0$ defines the hyperplane separating X_1 and X_2. Indeed, if $\alpha \geq 0$ and $x \in X_1$, then $\varphi(x, \alpha) \geq 0$ by (102) and, if $\alpha \leq 1$ and $x \in X_2$, then $\varphi \leq 0$ by (103). Thus, we obtain the parallel hyperplanes that strictly separate X_1 and X_2 for $0 < \alpha < 1$.

According to (88), the projection $\bar{x}^*$ of x^* onto the separating hyperplane (98) is calculated as

$$\bar{x}^* = x^* + A_1^\top z_1^* \left[z_1^{*\top}(b_1 - A_1 x^*) - \alpha \|z^*\|^2 \right] / \|A_1^\top z_1^*\|^2,$$

where $z_1^{*\top}(b_1 - A_1 x^*) = \|z_1^*\|^2$. Therefore, if α is given by (100), then $\bar{x}^* = x^*$; i.e., x^* belongs to the separating hyperplane (98).

Similarly, substituting the right-hand side of (100) for α^* in (99) and using the fact that $1 - \alpha^* = \|z_2^*\|^2/\|z^*\|^2$, we find that x^* belongs to the separating hyperplane (99).

Denote by $\mathrm{pr}\,(\alpha)$ the projection of the origin onto the hyperplane (98). By virtue of (88), we have

$$\mathrm{pr}\,(\alpha) = A_1^\top z_1^* \left[z_1^{*\top} b_1 - \alpha\|z^*\|^2\right] / \|A_1^\top z_1^*\|^2.$$

After simple calculations, we obtain $d = \|\,\mathrm{pr}\,(1) - \mathrm{pr}\,(0)\| = \|z^*\|^2/\|A_1^\top z_1^*\|$. The theorem is proved. □

The proof of assertion (i) of Theorem 6 is similar to that of Eremin's theorem [2, Theorem 10.1], which is based on a theorem of the alternative. In the notation adopted in this paper, the separating hyperplane in Eremin's theorem is described by the following equivalent equations:

$$u_1^{*\top}(A_1 x - b_1) + \rho/2 = 0, \qquad u_2^{*\top}(b_2 - A_2 x) - \rho/2 = 0,$$

where u_1^*, u_2^* is an arbitrary solution to the system

$$A_1^\top u_1 + A_2^\top u_2 = 0_n, \quad b_1^\top u_1 + b_2^\top u_2 = \rho > 0, \qquad u_1 \geq 0_k, \quad u_2 \geq 0_\ell. \tag{104}$$

By Theorem 6, to find a separating hyperplane, one must solve the problem of unconstrained minimization of the residual of the inconsistent system (I) in $\mathbb{R}^n$, whereas Eremin's theorem [2] implies that one must solve the consistent system (104) in m unknowns.

8. THE GORDAN AND STTEMKE THEOREMS OF THE ALTERNATIVE

The Gordan theorem of the alternative states that only one of the systems

$$Ax > 0_m, \tag{I$_8$}$$

$$A^\top u = 0_n, \qquad u \geq 0_m, \qquad \|u\|_1 > 0 \tag{II$_8$}$$

is solvable. This result does not follow directly from Theorem 3. By setting $b = \rho e_m$, where e_m is the m-dimensional unit vector, systems (I)$_6$ and (II)$_6$ are transformed into

$$Ax \geq \rho e_m, \tag{I$_8^*$}$$

$$A^\top u = 0_n, \quad u \geq 0_m, \quad \rho\|u\|_1 = \rho. \tag{II$_8^*$}$$

Systems (I)$_8$ and (I)$_8^*$ are solvable simultaneously; i.e., if system (I)$_8$ is solvable, then its solution x' determines the value of the parameter ρ equal to the minimal component of the vector Ax'. The vector x' corresponding to this value of ρ satisfies (I)$_8^*$. Vice versa, if system (I)$_8^*$ has a solution, then this solution obviously satisfies system (I)$_8$. Similarly, if u' solves system (II)$_8$, then $u = u'/\|u'\|_1$ is a solution to (II)$_8^*$. The converse is also true. Hence, the alternative in the Gordan theorem can be replaced by the alternative represented by systems (I)$_8^*$ and (II)$_8^*$, which define closed sets. Therefore, unlike (I)$_8$ and (II)$_8$, the alternative systems (I)$_8^*$ and (II)$_8^*$ can have normal solutions, which can be found by applying Theorem 3.

Note that both sides of the last equality in (II)$_8^*$ can be divided by ρ. However, in doing so, one must take into account the multiplier ρ^2 in the expression

$$\mathrm{pen}\,(x, X) = \sqrt{\|A^\top u\|^2 + \rho^2(1 - \|u\|_1)^2}$$

to ensure that (46) holds. Only under this condition can all results of Section 6 be extended to systems $(\mathrm{I})_8^*$ and $(\mathrm{II})_8^*$. In particular, formulas (94) and (95) become

$$\tilde{x}^* = \frac{A^\top u^*}{\rho(1 - \|u^*\|_1)}, \qquad \tilde{u}^* = \frac{\rho(\rho e_m - Ax^*)_+}{\|(\rho e_m - Ax^*)_+\|^2}.$$

The Stiemke theorem states that only one of the systems $Ax \geq 0_m$, $\|Ax\| > 0$, and $A^\top u = 0_n$, $u > 0_m$ is solvable.

As in the case of the Gordan theorem, the alternative systems in the Stiemke theorem can be replaced by systems that may have normal solutions. By introducing a vector of additional variables $\xi \in \mathbb{R}^m$, the first system in the Stiemke theorem is rewritten as

$$Ax - \xi = 0_m, \qquad \|\xi\|_1 = \rho, \qquad \xi \geq 0_m. \tag{I$_8^{**}$}$$

This system is solvable simultaneously with the first system in the Stiemke theorem. System $(\mathrm{II})_8^{**}$ is a special case of the general system (I) considered in Section 2. Therefore, its alternative is

$$A^\top u = 0_n, \quad -u + e_m\sigma \leq 0_m, \quad \rho\sigma = \rho > 0.$$

In a more compact form, it is written as

$$A^\top u = 0_n, \quad u \geq e_m. \tag{II$_8^{**}$}$$

System $(\mathrm{II})_8^{**}$ is solvable simultaneously with the second alternative system in the Stiemke theorem. Now, Theorem 3 can be applied to the alternative systems $(\mathrm{II})_8^{**}$ and $(\mathrm{II})_8^{**}$.

9. SYSTEMS OF LINEAR EQUATIONS IN NONNEGATIVE VARIABLES

Suppose that the system determining the set X has the form

$$Ax = b, \quad x \geq 0_n, \tag{I$_9$}$$

where A is an $m \times n$ matrix, $b \in \mathbb{R}^m$, and $\|b\| \neq 0$. The alternative system defining the set U has the form

$$A^\top u \leq 0_n, \quad b^\top u = \rho > 0, \tag{II$_9$}$$

where ρ is an arbitrary fixed constant.

By Theorem 3, only one of these systems, $(\mathrm{I})_9$ or $(\mathrm{II})_9$, is consistent. With the second relation in $(\mathrm{II})_9$ written as $b^\top u > 0$, this proposition is known as the Farkas lemma.

The vectors x^* and u^* are found by solving the problems

$$\min_{x \in \mathbb{R}^m_+} \|b - Ax\|^2/2 = \|b - Ax^*\|^2/2, \tag{105}$$

$$\min_{u \in \mathbb{R}^m} \left[\|(A^\top u)_+\|^2 + (\rho - b^\top u)^2\right]/2 = \left[\|(A^\top u^*)_+\|^2 + (\rho - b^\top u^*)^2\right]/2. \tag{106}$$

The necessary and sufficient minimality conditions for problems (105) and (106) are

$$-A^\top(b - Ax^*) \geq 0_n, \quad D(x^*)\left[A^\top(b - Ax^*)\right] = 0_n, \quad x^* \geq 0_n, \quad A(A^\top u^*)_+ - b(\rho - b^\top u^*) = 0_m.$$

Applying Theorems 1 – 3 and the results of Section 4, we obtain $b^\perp = z^* = b - Ax^*$, $b^\parallel = Ax^*$, and $w^{*\top} = [w_1^{*\top}, w_3^*]$, where $w_1^* = (A^\top u^*)_+$, and $w_3^* = \rho - b^\top u^*$.

If $X \neq \emptyset$, then $\|b^{\perp}\| = 0$, $U = \emptyset$, $w_3^* > 0$, and the normal solution to $(\mathrm{I})_9$ is expressed in terms of the solution to problem (106) as follows:

$$\tilde{x}^* = (A^\top u^*)_+/(\rho - b^\top u^*).$$

Using the condition $\tilde{x}^* \in X$, we obtain an analogue of formula (75):

$$A(A^\top u^*)_+ + bb^\top u^* = \rho b.$$

If $\|b^{\perp}\| \neq 0$, then $X = \emptyset$, $U \neq \emptyset$, $\|z^*\| \neq 0_m$, and the normal solution to $(\mathrm{II})_9$ has the form

$$\tilde{u}^* = \rho(b - Ax^*)/\|b - Ax^*\|^2 = \rho b^{\perp}/\|b^{\perp}\|^2.$$

By applying the optimal correction procedure, the inconsistent system $(\mathrm{I})_9$ is reduced to $Ax = b^{\|}$, $x \geq 0_n$. Then, the alternative inconsistent system is written as $A^\top u \leq 0_n$, $x^{*\top} A^\top u = \rho > 0$.

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