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# THE AUGMENTED LAGRANGIAN FUNCTION FOR THE LINEAR PROGRAMMING PROBLEM[1]

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**1.** The augmented Lagrangian function is used to solve the linear programming (LP) problem. We define the interval of the penalty coefficient that provides a means of finding a solution of the original problem via a single minimization of a quadratic function on the positive orthant.

Consider the primal and dual linear programming problems:

$$f_* = \min_{x \in X} c^\top x, \qquad X = \{x \in \mathbb{R}^n_+ : b - Ax = 0_m\}, \tag{1}$$

$$\max_{u \in U} b^\top u, \qquad U = \{u \in \mathbb{R}^m : c - A^\top u \geq 0_n\}. \tag{2}$$

Here and further, $A$ denotes an $m \times n$ matrix with rank $m$, $m < n$; vectors $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, $x \in \mathbb{R}^n_+$, $u \in \mathbb{R}^m$, $0_i$ is the $i$-dimensional null vector.

Let

$$H(x, u, \varepsilon) = c^\top x + u^\top (b - Ax) + \frac{\|b - Ax\|^2}{2\varepsilon}$$

denote the augmented Lagrangian function, where $\varepsilon > 0$ is the penalty coefficient and $\|\cdot\|$ is the Euclidean norm. Consider the auxiliary minimization problem

$$\min_{x \in \mathbb{R}^n_+} H(x, \tilde{u}, \varepsilon). \tag{3}$$

Here the Lagrange multiplier $\tilde{u}$ is fixed. If $\tilde{u} \equiv 0_m$, then the function $H(x, 0, \varepsilon)$ is the well-known quadratic penalty function. Therefore, the results obtained bellow are applicable to the traditional exterior method based on the quadratic penalty function.

We assume that the primal LP problem (1) may have a unique solution $x_*$ which may be degenerate. Let $x_*^L > 0_\ell$ denote a set of positive components of vector $x_*$. For a nondegenerate solution $x_*$ we have $\ell = m$. We denote by $I_*^L$ the set of these positive components of vector $x_*$. If $x_*$ is the unique degenerate solution, then the dual LP problem (2) has multiple solutions. From the solution set $U_*$ of the dual LP problem (2) we can choose a vector $u_*$ which is the nearest to some fixed vector $\tilde{u}$. Therefore, the vector $u_*$ is the unique solution of the following quadratic problem:

$$\min \left\{ \frac{\|u - \tilde{u}\|^2}{2} : c - A^\top u \geq 0_n, \quad b^\top u \geq f_* \right\}. \tag{4}$$

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The pair $[x_*, u_*]$ satisfies the Kuhn–Tucker conditions for problem (1) and these conditions can be written as follows:

$$\begin{array}{rclr} v_*^L & = & c^L - B_L^\top u_* = 0_\ell, & x_*^L > 0_\ell; \\ v_*^S & = & c^S - B_S^\top u_* = 0_s, & x_*^S > 0_s; \\ v_*^N & = & c^N - N^\top u_* > 0_r, & x_*^N = 0_r; \\ B_L x_*^L & = & b. & \end{array} \tag{5}$$

Here the matrix $A = [B \vdots N]$ corresponds to partitioning the vector $v_* = c - A^\top u_*$ into zero $\left[[v_*^L]^\top, [v_*^S]^\top\right] = [v_*^B]^\top = 0_k$ and positive $v_*^N > 0_r$ components, where $k = \ell + s \leq m$, $r = n - k$. Using this partition, the vector $x_*$ can be represented as $[x_*]^\top = \left[[x_*^B]^\top, [x_*^N]^\top\right]$. The matrix $B$ corresponds to partitioning the vector $x_*^B$ into positive $x_*^L > 0_\ell$ and zero $x_*^S = 0_s$ components, i.e., $B = [B_L \vdots B_S]$. Since the solution $x_*$ is unique, the matrix $B$ consists of $k \leq m$ linearly independent columns, i.e., its rank is $k$.

Let us introduce a vector $\eta \in \mathbb{R}^k$ as $\eta = (B^\top B)^{-1}(c^B - B^\top \tilde{u})$. Notice that if $k = m$, then $\eta = B^{-1}(u_* - \tilde{u})$.

**Lemma 1.** *The solution of problem* (4) *is given by the vector*

$$u_* = \tilde{u} + B\eta = \tilde{u} + B_L \eta^L + B_S \eta^S,$$

*where* $\eta^S \leq 0_s$.

**Proof**. Since the vector $u_*$ is a unique solution of problem (4), there exist Lagrange multipliers $\bar{x} \geq 0_n$, $\bar{t} \geq 0$, such that the triple $[u_*, \bar{x}, \bar{t}]$ satisfies the Kuhn–Tucker conditions for problem (4):

$$\begin{array}{c} D(\bar{x}) v_* = 0_n, \qquad v_* = c - A^\top u_* \geq 0_n, \qquad \bar{x} \geq 0_n, \\ u_* - \tilde{u} - \bar{t} b + A\bar{x} = 0_m, \\ b^\top u_* = f_*, \qquad \bar{t} \geq 0. \end{array} \tag{6}$$

Since the pair $[x_*, u_*]$ satisfies the Kuhn–Tucker condition (5), it is possible to rewrite (6) as

$$\begin{array}{rclr} v_*^B & = & c^B - B^\top u_* = 0_k, & \bar{x}^B \geq 0_k, \\ v_*^N & = & c^N - N^\top u_* > 0_r, & \bar{x}^N = 0_r, \end{array} \tag{7}$$

$$\begin{array}{c} u_* - \tilde{u} - \bar{t} b + B\bar{x}^B = 0_m, \\ b^\top u_* = f_*, \qquad \bar{t} \geq 0. \end{array} \tag{8}$$

Solving the system of equations (7) and (8), we have

$$\begin{array}{c} \bar{x}^B = \bar{t}(B^\top B)^{-1} B^\top b - \eta \geq 0_k, \\ u_* = \tilde{u} + \bar{t} M b + B\eta. \end{array} \tag{9}$$

Here we denote the orthogonal projector by $M = I - B(B^\top B)^{-1} B^\top$ and, since the vector $b$ belongs to the column space of matrix $B^L$, $Mb = 0_n$ holds. The matrix $B$ consists of linearly independent columns, hence there exists only one vector $\eta = \begin{bmatrix} \eta^L \\ \eta^S \end{bmatrix}$ such that $u_* - \tilde{u} = B\eta = B_L \eta^L + B_S \eta^S$. We show that $\eta^S \leq 0_s$. From the condition $Bx_*^B = b$ we obtain

$$x_*^B = (B^\top B)^{-1} B^\top b = \begin{bmatrix} x_*^L \\ x_*^S \end{bmatrix} \begin{array}{l} > 0_\ell \\ = 0_s \end{array}.$$

From (9) and the last formula, we have

$$\bar{x}^B = \begin{bmatrix} \bar{x}^L \\ \bar{x}^S \end{bmatrix} = \begin{bmatrix} \bar{t}x_*^L - \eta^L \\ -\eta^S \end{bmatrix} \geq 0_k. \tag{10}$$

This concludes the proof of the lemma. $\square$

Further, we introduce the following penalty coefficient:

$$\varepsilon_* = \begin{cases} \min\limits_{i \in I_*^L,\, (\eta^L)^i > 0} \dfrac{(x_*^L)^i}{(\eta^L)^i}; \\ +\infty, \qquad \text{if } (\eta^L)^i \leq 0 \text{ for all } i \in J_*^L. \end{cases} \tag{11}$$

**Theorem 1.** *Let $x_*$ be a unique (possibly degenerate) solution of the LP problem* (1)*. Then for each $0 < \varepsilon \leq \varepsilon_*$, where $\varepsilon_*$ is defined by* (11)*, we have:*

**1)** *the problem* (3) *has a unique solution $x(\varepsilon)$ with components: $x^L(\varepsilon) = x_*^L - \varepsilon\eta^L \geq 0_\ell$, $x^S(\varepsilon) = -\varepsilon\eta^S \geq 0_s$ and $x^N(\varepsilon) = 0_r$;*

**2)** *the solution of the dual LP problem* (2) *is given by*

$$u_* = \bar{u} + \frac{[b - Ax(\varepsilon)]}{\varepsilon}. \tag{12}$$

**Proof.** The necessary and sufficient conditions for a minimum of problem (3) can be formulated as follows:

$$\varepsilon(c^L - B_L^\top \tilde{u}) - B_L^\top(b - B_L x^L(\varepsilon) - B_S x^S(\varepsilon)) = 0_\ell, \qquad x^L(\varepsilon) \geq 0_\ell, \tag{13}$$

$$\varepsilon(c^S - B_S^\top \tilde{u}) - B_S^\top(b - B_L x^L(\varepsilon) - B_S x^S(\varepsilon)) = 0_s, \qquad x^S(\varepsilon) \geq 0_s, \tag{14}$$

$$\varepsilon(c^N - N^\top \tilde{u}) - N^\top(b - B_L x^L(\varepsilon) - B_S x^S(\varepsilon)) = 0_r, \qquad x^N(\varepsilon) \geq 0_r. \tag{15}$$

Since the $m \times k$ matrix $B = [B_L \vdots B_S]$ has rank $k$, it is easy to show that the system of equations (13), (14) has a unique solution

$$x^B(\varepsilon) = \begin{bmatrix} x^L(\varepsilon) \\ x^S(\varepsilon) \end{bmatrix} = \begin{bmatrix} x_*^L - \varepsilon\eta^L \\ -\varepsilon\eta^S \end{bmatrix} \geq 0_k. \tag{16}$$

Substituting $x^B(\varepsilon)$ into (15) yields $\varepsilon(c^N - N^\top\tilde{u}) - N^\top(b - B_L x_*^L + \varepsilon B_L \eta^L + \varepsilon B_S \eta^S) = \varepsilon(c^N - -N^\top(\tilde{u} + B\eta)) = \varepsilon(c^N - N^\top u_*) = \varepsilon v_*^N > 0_r$. The last inequality holds for any positive $\varepsilon$.

Inequality (16) holds due to the lemma's assertion ($\eta^S \leq 0_s$) and the existence of the interval $(0, \varepsilon]$ mentioned in the condition of the theorem. Hence, the system (13) – (15) is consistent and statement **1)** is proved.

Proof of assertion **2)** follows from substituting the solution $x(\varepsilon)$ into the expression (12). $\square$

**Remark 1.** *Note the important special case where the problem* (1) *has a nondegenerate solution $x_*$. In this case the indices of nonzero components of vectors $x(\varepsilon)$ and $x_*$ coincide for each $0 < \varepsilon < \varepsilon_*$. Since $x_*^B = x^B(\varepsilon) + \eta$, we obtain a means of solving the problem* (1) *via a single minimization of $H(x, \tilde{u}, \varepsilon)$ on the positive orthant. Hence, $H(x, \tilde{u}, \varepsilon)$ could be called "the exact auxiliary function" if we slightly extend the notion defined in* [1].

**Remark 2.** *Formulas* (10) *and* (16) *show that there exists a certain connection between the solution $x(\varepsilon)$ of problem* (3) *and the optimal Lagrange multiplier $\bar{x}$ for problem* (4)*. Expression*

(10) *is valid for* $\bar t \geq \dfrac{1}{\varepsilon_*}$. *If we set* $\bar t = \dfrac{1}{\varepsilon}$, *then from* (10) *and* (16) *we obtain that for each* $0 < \varepsilon \leq \varepsilon_*$ *the equality* $x^B(\varepsilon) = \varepsilon \bar x^B$ *holds.*

$\boxed{\textbf{2.}}$ It is well known (see, e.g., [2]), that problem (3) is dual to the following problem:

$$\max_{u \in U} \left[ b^\top u - \varepsilon \frac{\|u - \tilde u\|}{2} \right], \qquad U = \{u \in \mathbb{R}^m : c - A^\top u \geq 0_n\}. \tag{17}$$

We denote by $u(\varepsilon)$ the solution of this problem. Then the solutions $x(\varepsilon)$ and $u(\varepsilon)$ are connected by

$$u(\varepsilon) = \tilde u + \frac{b - Ax(\varepsilon)}{\varepsilon}.$$

The following theorem is a direct consequence of Theorem 1 due to the above duality.

**Theorem 2.** *Let the conditions of theorem* 1 *hold. Then a unique solution of problem* (17) *does not depend on* $\varepsilon$ *and coincides with the vector* $u_*$, *that is a solution of the problem* (2).

**Remark 3.** *Paper* [3] *was probably the first where the coincidence of the solution of problem* (17) *and the solution of problem* (2) *was investigated (in the case* $\tilde u \equiv 0_m$*). Various estimates of value* $\varepsilon_*$, *using the Lagrange multiplier* $t$, *for problem* (4) *were obtained in* [4, 5, 6] *for the same case. The result concerning the finiteness of the augmented Lagrangian method (problem* (3)*) was first obtained in* [7] *under the assumption of solvability of the LP problem. This question was thoroughly investigated in* [8] *with the help of the sensitivity function.*

## References


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