ISSN 0005-1179, Automation and Remote Control, 2014, Vol. 75, No. 6, pp. 1025–1040. © Pleiades Publishing, Ltd., 2014.
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Method of Non-uniform Coverages to Solve the Multicriteria Optimization Problems with Guaranteed Accuracy

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Received April 1, 2013

Abstract—Application of the non-uniform coverage method to the multicriteria optimization problems was considered, and the concept of the ε-Pareto set was formulated and studied. An algorithm to construct a ε-Pareto set with a guaranteed accuracy ε was described. Efficient implementation of this approach was described, and the results of experiments were presented.

DOI: 10.1134/S0005117914060046

1. INTRODUCTION

Nowadays numerous studies on the multicriteria optimization have been published. For the information about the methods to solve such problems the readers are referred to the monographs [1–3]. A detailed review of the state-of-the-art in such studies can be found in [4]. The methods based on approximation of the Pareto set [2, 5–10] represent one of the important lines of research in the multicriteria optimization. There are also other approaches to the problem of multicriteria optimization that are described in [12–15].

The present paper considers an approach to approximation of the Pareto set on the basis of the method of non-uniform coverage. This method was used to advantage for seeking the global extremum of multivariable functions. Its development, generalization, and efficient realization were discussed in numerous publications of which we mention the pioneering book [11] and some of the latest papers [16–20]. Owing to a successful definition of the notion of the ε-Pareto set, the method was extended to the multicriteria problems [8].

The method of non-uniform coverages enables one to establish the ε-Pareto set for the given ε, that is, guarantees the ε-optimality of the resulting approximation. This distinction of the method of non-uniform coverage is unique and not met in other approaches to the problem of multicriteria optimization. For some applications of the multicriteria optimization problems such as construction of the reachability boundary of the manipulator robot [21], it is very important to have a guaranteed accuracy of solution.

The present paper proves new properties of the ε-Pareto set and demonstrates its relation to the Edgeworth–Pareto hull. The initial variant of the method proposed in [8] was used assuming that the criteria function satisfied the Lipschitz condition. The method of non-uniform coverage for the multicriteria problems is generalized below to the case of arbitrary minorants. Efficiency of the proposed approach is demonstrated by comparative analysis of the numerical calculations of two simplest problems.

The following notation is used below. The components of the n-dimensional vector x are denoted by the parenthesized superscript as $x = (x^{(1)}, \ldots, x^{(n)})$. The Euclidean norm in the space $\mathbb{R}^n$ is denoted by $\|x\|$: $\|x\| = \sqrt{\sum_{i=1}^n (x^{(i)})^2}$. The vector $(|x^{(1)}|, \ldots, |x^{(n)}|)$ consisting of the absolute values of the components of x is denoted by $|x|$. The vector inequalities are satisfied componentwise.

2. DEFINITION OF THE PARETO-OPTIMAL SOLUTION OF THE MULTICRITERIA PROBLEM

The problem of multicriteria minimization is denoted conventionally as

$$\min_{x \in X} F(x), \tag{1}$$

where X is the permissible parameter set and the vector function $F(\cdot) : \mathbb{R}^n \to \mathbb{R}^m$ defines the vector criterion whose components $f^{(1)}(\cdot), \ldots, f^{(m)}(\cdot)$ make up a collection of m scalar criteria. The image $Y = F(X)$ of the permissible set X under the map F is called the *set of reachable criterial vectors*. In what follows, the vector function $F(\cdot)$ is assumed to be continuous, and the set X, nonempty compact. Under these assumptions, Y is also a nonempty compact set.

For an arbitrary point $z \in \mathbb{R}^m$, we define the *southwest* $\mathrm{SW}(z)$ and *northeast* $\mathrm{NE}(z)$ sets as

$$\mathrm{SW}(z) = \{y \in \mathbb{R}^m : y \leqslant z\}, \quad \mathrm{NE}(z) = \{y \in \mathbb{R}^m : y \geqslant z\}.$$

For an arbitrary set $\Omega \subseteq \mathbb{R}^m$, we define its *Pareto set* $\mathcal{P}(\Omega)$ as

$$\mathcal{P}(\Omega) = \{\omega \in \Omega : \Omega \cap \mathrm{SW}(\omega) = \omega\}. \tag{2}$$

If the set of reachable criterial vectors Y is taken as Ω, then this definition coincides with the standard definition of the Pareto set used in the publications on multicriteria optimization.

The following relation which is valid for any $\Omega \subseteq \mathbb{R}^m$ is true for the so-defined map $\mathcal{P} : 2^{\mathbb{R}^m} \to 2^{\mathbb{R}^m}$:

$$\mathcal{P}\left(\mathcal{P}(\Omega)\right) = \mathcal{P}(\Omega) \subseteq \Omega. \tag{3}$$

We expand the definitions of $\mathrm{SW}(z)$ and $\mathrm{NE}(z)$ to the case of arbitrary set $\Omega \subseteq \mathbb{R}^m$:

$$\mathrm{SW}(\Omega) = \cup_{y \in \Omega} \mathrm{SW}(y), \quad \mathrm{NE}(\Omega) = \cup_{y \in \Omega} \mathrm{NE}(y).$$

The set $\mathrm{NE}(\Omega)$ is called the *hull of the Edgeworth–Pareto set* Ω. Validity of the following property which is true for any nonempty compact set Ω in the space $\mathbb{R}^m$ can be readily demonstrated:

$$\Omega \subseteq \mathrm{NE}\left(\mathcal{P}(\Omega)\right) = \mathrm{NE}(\Omega). \tag{4}$$

Solution of problem (1) lies in determining the set $P(Y)$ and its preimage under the map F. The above definitions do not constrain the cardinality of the set X. It may be continual, countable, or finite. Since Y is a nonempty compact, $P(Y)$ is not empty [3].

3. NOTION OF THE ε-PARETO SET

We follow [8] in defining the notion of approximate solution of problem (1). For $\varepsilon \geqslant 0$, the discrete set of points $Y_\varepsilon \subseteq Y$ is called the ε-*Pareto set* if

$$\text{for any point } y_* \in \mathcal{P}(Y) \text{ there exists a point } y_\varepsilon \in Y_\epsilon \text{ such that } y_\varepsilon - \varepsilon \times e_m \leqslant y_*, \tag{5}$$

and

$$\mathcal{P}(Y_\varepsilon) = Y_\varepsilon. \tag{6}$$

Here, the vector e_m denotes a vector from the space $\mathbb{R}^m$ with all components equal to 1. Relations (5) and (6) are called, respectively, the *first and second conditions for ε-Pareto optimality.* The second condition enables one to reject the excessive points from the discrete point collection Y_ε. The set $A_\varepsilon \subseteq X$ such that $F(A_\varepsilon) = Y_\varepsilon$ is called the *ε-optimal solution* of problem (1) which is assumed to be solved with the given accuracy ε if the ε-Pareto set Y_ε and its preimage A_ε are determined.

Lemma 1. *If the ε-Pareto set Y_ε is constructed, then for any point y from the Edgeworth–Pareto* $\mathrm{NE}(Y)$ *hull there exists a point $y_\varepsilon \in Y_\varepsilon$ such that*

$$y_\varepsilon - \varepsilon e_m \leqslant y. \tag{7}$$

Proof. Let $y \in \mathrm{NE}(Y)$. According to (4), there exists a point $y_* \in \mathcal{P}(Y)$ such that $y_* \leqslant y$. According to (5), for the point y_* a point y_ε will turn up such that $y_\varepsilon - \varepsilon e_m \leqslant y_*$. Therefore, $y_\varepsilon - \varepsilon e_m \leqslant y_* \leqslant y$, and, consequently, inequality (7) is true.

It follows from (5) that the ε-Pareto set is at the same time the $\overline{\varepsilon}$-Pareto set for any $\overline{\varepsilon}$, $\overline{\varepsilon} \geqslant \varepsilon$. In particular, $\mathcal{P}(Y)$ is the ε-Pareto set for any $\varepsilon \geqslant 0$. Consequently, under the above assumptions about the compactness and nonemptiness of Y for any $\varepsilon \geqslant 0$ there always exists at least one ε-Pareto set. The ε-Pareto set is defined uniquely only for $\varepsilon = 0$. In this case, it coincides with $\mathcal{P}(Y)$. Generally speaking, for $\varepsilon > 0$ there may be arbitrarily many sets meeting the introduced definition.

The boundary of the set $\mathrm{NE}(Y)$ is denoted by $\partial(\mathrm{NE}(Y))$. We consider the set

$$S_\varepsilon(Y) = Y \cap \cup_{y \in \partial(\mathrm{NE}(Y))} \mathrm{SW}(y + \varepsilon e_m)$$

which is called below the ε-band of the set Y.

Assertion 1. *The inclusion*

$$Y_\varepsilon \subseteq S_\varepsilon(Y) \tag{8}$$

is valid for any ε-Pareto set Y_ε.

Proof. Let inclusion (8) be not satisfied, which means that there exists ε-Pareto set Y_ε comprising a point y_ε not belonging to the ε-band. It is easy to prove that in this case the point $y = y_\varepsilon - \varepsilon e_m$ is the interior point of the set $\mathrm{NE}(Y)$. According to (4), there exists a point y_* from $\mathcal{P}(Y)$ such that $y_* \leqslant y$. Since y_* is the boundary point of Y and y is the interior point, $y_* \neq y$. By the definition of the ε-Pareto set, there exists a point $v \in Y_\varepsilon$ such that

$$v - \varepsilon e_m \leqslant y_* \leqslant y = y_\varepsilon - \varepsilon e_m.$$

Whence it follows that $v \leqslant y_\varepsilon$. At that, $v \neq y_\varepsilon$, and we encounter a contradiction with the second optimality condition $\mathcal{P}(Y_\varepsilon) = Y_\varepsilon$.

For any point $u \in \mathbb{R}^m$ and the set $V \subseteq \mathbb{R}^m$, we define the distance from u to V as $\rho(u, V) = \inf_{v \in V} \|u - v\|$.

By the *deviation of the nonempty set $U \subseteq \mathbb{R}^m$ from the nonempty set $V \subseteq \mathbb{R}^m$* is meant the value

$$d(U, V) = \sup_{u \in U} \rho(u, V).$$

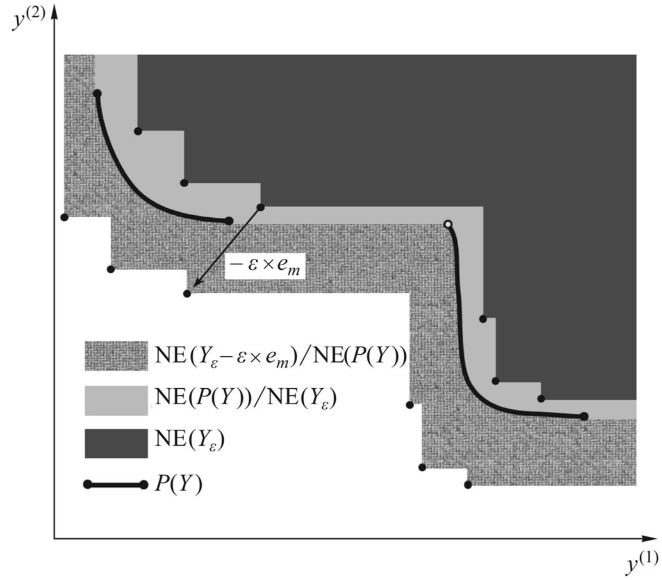

Fig. 1. Illustration of Theorem 1.

The deviation $d(U, V)$ is not symmetrical relative to the permutation of the arguments because, generally speaking, the values of $d(U, V)$ and $d(V, U)$ differ. We introduce a symmetrical *Hausdorff distance* between two nonempty subsets U and V of the space $\mathbb{R}^m$:

$$d_H(U, V) = \max\left(d(U, V), d(U, V)\right).$$

Theorem 1. *Under the above assumptions about the set Y, valid are the following assertions relating the Edgeworth–Pareto hulls*[1] *of the ε-Pareto set and the set of reachable criteria vectors*

$$\mathrm{NE}(Y_\varepsilon) \subseteq \mathrm{NE}(\mathcal{P}(Y)) = \mathrm{NE}(Y) \subseteq \mathrm{NE}(Y_\varepsilon - \varepsilon \times e_m), \tag{9}$$

$$d_H(\mathrm{NE}(Y_\varepsilon), \mathrm{NE}(Y)) \leqslant d_H(\mathrm{NE}(Y_\varepsilon), \mathrm{NE}(Y_\varepsilon - \varepsilon \times e_m)) \leqslant \varepsilon, \tag{10}$$

$$d_H(\mathrm{NE}(Y), \mathrm{NE}(Y_\varepsilon - \varepsilon \times e_m)) \leqslant d_H(\mathrm{NE}(Y_\varepsilon), \mathrm{NE}(Y_\varepsilon - \varepsilon \times e_m)) \leqslant \varepsilon. \tag{11}$$

The proof of these assertions that are illustrated in Fig. 1 is evident.

Theorem 1 implies that as ε tends to zero, the sets $\mathrm{NE}(Y_\varepsilon)$ and $\mathrm{NE}(Y_\varepsilon - \varepsilon \times e_m)$ tend in the Hausdorff metric to the Edgeworth–Pareto hull $\mathrm{NE}(Y)$ of the set of reachable criterial vectors inside and outside the set Y (Fig. 1). At that, the set $\mathcal{P}(Y)$ lies in between the sets $\mathrm{NE}(Y_\varepsilon - \varepsilon \times e_m)$ and $\mathrm{NE}(Y_\varepsilon)$, that is,

$$\mathcal{P}(Y) \in (\mathrm{NE}(Y_\varepsilon - \varepsilon \times e_m) \setminus \mathrm{NE}(Y_\varepsilon)) \cup Y_\varepsilon.$$

Now we relate the ε-Pareto set and the Pareto set for problem (1).

Lemma 2. *Let $y_* \in \mathcal{P}(Y)$. Then, for any $\delta > 0$ there is $\varepsilon > 0$ such that the inequality*

$$\rho(y_*, Y_\varepsilon) \leqslant \delta \tag{12}$$

is satisfied for the ε-Pareto set Y_ε.

Proof. Let us consider a monotone decreasing sequence $\{\varepsilon_k\}$, $\varepsilon_k > 0$, $\lim_{k\to\infty} \varepsilon_k = 0$ tending to zero. We assume that the lemma is invalid. Then, for any k there exists a ε_k-Pareto set Y_{ε_k}

[1] The idea of using the Edgeworth–Pareto hull in the solution of the multicriteria problems belongs to A.V. Lotov [22].

such that $\rho(y_*, Y_{\varepsilon_k}) > \delta$. By definition of the ε-Pareto set, there exists a point $y_k \in Y_{\varepsilon_k}$ such that $y_k - \varepsilon_k e_m \leqslant y_*$. By virtue of compactness of the set Y, we can assume without loss of generality that the sequence $\{y_k\}$ converges to the point $y \in Y$. Since $\lim_{k\to\infty} \varepsilon_k = 0$ and $y_k - \varepsilon_k e_m \leqslant y_*$, we have $y \leqslant y_*$. Since $\rho(y_*, Y_{\varepsilon_k}) > \delta$ for each k, we have $\rho(y_*, y) \geqslant \delta$ and, consequently, $y_* \neq y$, which contradicts to the fact that y_* belongs to $\mathcal{P}(Y)$.

We cite a definition from [23]. Let $\varepsilon > 0$ be a given positive number. The set $\Omega \subseteq \mathbb{R}^m$ is called the *ε-network* for the set $Y \subseteq \mathbb{R}^m$ if for each point $y \in Y$ there exists a point z from Ω such that $\|y - z\| \leqslant \varepsilon$.

Theorem 2. *For any $\delta > 0$, there exists $\varepsilon > 0$ such that any ε-Pareto set Y_ε makes up a δ-network for the set $\mathcal{P}(Y)$.*

Proof. Obviously, the set Y_ε makes up the ε-network of the set $\mathcal{P}(Y)$ if and only if

$$d(\mathcal{P}(Y), Y_\varepsilon) \leqslant \delta.$$

We consider the monotone sequence $\{\varepsilon_k\}$, $\varepsilon_k > 0$, $\lim_{k\to\infty} \varepsilon_k = 0$ tending to zero and assume that the theorem is invalid. Then, for any k there exists a ε_k-Pareto set Y_{ε_k} such that $d(\mathcal{P}(Y), Y_{\varepsilon_k}) > \delta$, which implies existence of the point $y_k \in \mathcal{P}(Y)$ such that $\rho(y_k, Y_{\varepsilon_k}) > \delta$. In virtue of compactness of the set Y, we can assume without loss of generality that the sequence $\{y_k\}$ converges to some point $y \in Y$. Let K be a natural number such that $\|y_i - y\| \leqslant \delta/3$ for $i \geqslant K$.

According to Lemma 2, there exists $\varepsilon > 0$ such that $\rho(y_K, Y_\varepsilon) \leqslant \delta/3$ is valid for any ε-Pareto set Y_ε. Let N be an integer greater than or equal to K such that $\varepsilon_N \leqslant \varepsilon$. According to the assumption, $\rho(y_N, Y_{\varepsilon_N}) > \delta$. On the other hand, since Y_{ε_N} is the ε-Pareto set, $\rho(y_K, Y_{\varepsilon_N}) \leqslant \delta/3$. Consequently, there exists a point $u \in Y_{\varepsilon_N}$ such that $\|u - y_K\| \leqslant \delta/3$. According to the properties of the norm, we get $\|u - y_N\| \leqslant \|u - y_K\| + \|y_K - y\| + \|y - y_N\| \leqslant \delta$ and encounter contradiction with the inequality $\rho(y_N, Y_{\varepsilon_N}) > \delta$, which proves the theorem.

Theorem 2 does not relate ϵ and δ. Such relation may be established for the Jeoffrion-optimal points. We introduce notation for the set of indices of the criteria $M = \{1, \ldots, m\}$. According to [3], $y_* \in \mathcal{P}(Y)$ is called the *Jeoffrion-optimal* point if there exists a positive number $\theta(y_*)$ such that for all points $y \in Y$ the following is valid: if $y^{(i)} < y_*^{(i)}$, then there exists $j \in M$ such that $y^{(j)} > y_*^{(j)}$ and

$$\frac{y_*^{(i)} - y^{(i)}}{y^{(j)} - y_*^{(j)}} \leqslant \theta(y_*). \tag{13}$$

Assertion 2. *If y_* is the Jeoffrion-optimal point, then for any ε-Pareto set Y_ε the inequality*

$$\rho(y_*, Y_\varepsilon) \leqslant \varepsilon\sqrt{m}\max(1, \theta(y_*)) \tag{14}$$

is valid for $\varepsilon > 0$.

Proof. Let $\varepsilon > 0$ and Y_ε be the ε-Pareto set. By definition of the ε-Pareto set, there exists a point $y_\varepsilon \in Y_\varepsilon$ such that $y_\varepsilon - \varepsilon \times e_m \leqslant y_*$. We define two subsets M_+ and M_- of the set M:

$$M_+ = \left\{i \in M : y_*^{(i)} > y_\varepsilon^{(i)}\right\},$$
$$M_- = \left\{i \in M : y_*^{(i)} \leqslant y_\varepsilon^{(i)}\right\}.$$

Obviously, $M = M_+ \cup M_-$. Since $y_\varepsilon - \varepsilon e_m \leqslant y_*$, we get that $0 \leqslant y_\varepsilon^{(i)} - y_*^{(i)} \leqslant \varepsilon$ is valid for any $i \in M_-$. Since y is Jeoffrion-optimal, for any $i \in M_+$ there exists $j \in M_-$ such that

$$\frac{y_*^{(i)} - y_\varepsilon^{(i)}}{y_\varepsilon^{(j)} - y_*^{(j)}} \leqslant \theta(y_*).$$

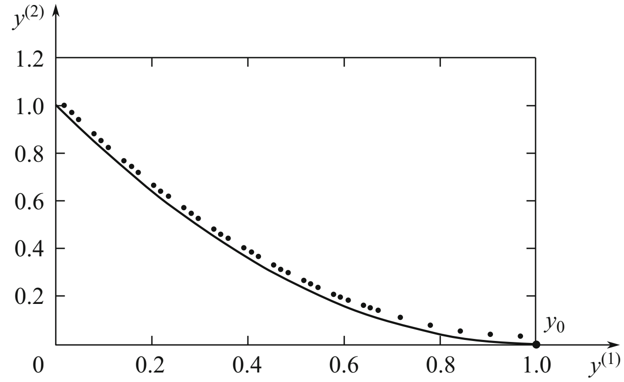

Fig. 2. Illustrative example for Theorem 2.

Consequently,

$$0 \leqslant y_*^{(i)} - y_\varepsilon^{(i)} \leqslant \theta(y_*)(y_\varepsilon^{(j)} - y_*^{(j)}) \leqslant \theta(y_*)\varepsilon$$

is satisfied for $i \in M_+$. Therefore, the inequality $|y_\varepsilon^{(i)} - y_*^{(i)}| \leqslant \delta$, where $\delta = \max(\varepsilon, \theta(y_*)\varepsilon)$, is valid for all $i \in M$. The following theorem follows from the definition of the Euclidean metric.

Assertion 2 enables one to estimate from above the accuracy of approximation provided by the ε-Pareto set with the use of $\theta(y)$. We consider problem (2) where the criteria and the permissible set are defined as follows:

$$\begin{gathered} f^{(1)}(x) = x^{(1)}, \\ f^{(2)}(x) = (x^{(1)} - 1)^2 + x^{(2)}, \\ 0 \leqslant x^{(1)} \leqslant 1, \\ 0 \leqslant x^{(2)} \leqslant 1. \end{gathered} \tag{15}$$

The bold line in Fig. 2 represents the Pareto set defined in the criteria space by a segment of the curve $y^{(2)} = (y^{(1)} - 1)^2$. All points of this set, except for $y_0 = (1, 0)$, are Jeoffrion-optimal. At that,

$$\theta(y_*) = \max\left(2(1 - y^{(1)}), \frac{1}{2(1 - y^{(1)})}\right),$$

where $y_* = (y^{(1)}, (y^{(1)} - 1)^2)$.

The marked points make up the ε-Pareto set obtained for $\varepsilon = 0.05$ by the method of non-uniform coverages. One can see that in the neighborhood of the point y_0 where the approaching parameter $\theta(y_*)$ grows without limit the approximation accuracy is smaller than at other points. This property corresponds to the estimate (14). The maximal density of the points of the ε-Pareto set is observed at the central part of the graph where $\theta(y_*)$ is minimal.

The accuracy of discrete approximation over the given segment comprising only the Jeoffrion point can also be estimated numerically. For example, over the segment $f^{(1)} \in [0, 1/2]$ the parameter $\theta(y_*)$ does not exceed 2. Consequently, on this segment the distance from any point of the Pareto set to the nearest point of the ε-Pareto set does not exceed $2\sqrt{2}\varepsilon$.

4. SEARCH OF AN APPROXIMATE SOLUTION TO THE MULTICRITERIA PROBLEM

4.1. General Theory

We recall the fundamental idea of the method of non-uniform coverages for the single-criterion problems. The problem of seeking the global minimum on the compact *permissible set* $X \subseteq \mathbb{R}^n$ is formulated for the continuous scalar function $f : \mathbb{R}^n \to \mathbb{R}$:

$$f_* = glob \min_{x \in X} f(x) = f(x_*), \tag{16}$$

where x_* is any point where the global minimum f_* is reached. For this problem, we define the set of *global decisions* X_* and the set of ε-*optimal decisions* X_ε:

$$X_* = \{x \in X : f(x) = f_*\}, \quad X_\varepsilon = \{x_\varepsilon \in X : f(x_\varepsilon) \leqslant f_* + \varepsilon\}, \quad \varepsilon > 0. \tag{17}$$

We assume that the set X_* is nonempty. For approximate solution of problem (16), it suffices to determine at least one point x_ε of the set X_ε. Then, $f(x_\varepsilon)$ exceeds f_* at most by ε.

For the function $f(x) : \mathbb{R}^n \to \mathbb{R}$, set $\Omega \subseteq \mathbb{R}^n$, and number $\lambda \in \mathbb{R}$ we define the *Lebesgue set*

$$\mathcal{L}(f(\cdot), \Omega, \lambda) = \{x \in \Omega : f(x) \geqslant \lambda\}. \tag{18}$$

Using the notion of the Lebesgue set, it is possible to define the necessary and sufficient conditions for global optimality of the point $x_* \in X$ as $x_* \in X_* \Leftrightarrow \mathcal{L}(f(\cdot), X, f(x_*)) = X$.

The criterion for global ε-optimality is set down in a similar way. Let $x_\varepsilon \in X$, then

$$x_\varepsilon \in X_\varepsilon \Leftrightarrow \mathcal{L}(f(\cdot), X, f(x_\varepsilon) - \varepsilon) = X. \tag{19}$$

Now we consider the case of m criteria where $m > 1$. For an arbitrary set $\Lambda \subseteq Y$, $\Omega \subseteq \mathbb{R}^n$, and the vector function $F(\cdot) : \mathbb{R}^n \to \mathbb{R}^m$, we define the *Lebesgue set*

$$\mathcal{L}(F(\cdot), \Omega, \Lambda) = \{x \in \Omega : F(x) \in \mathrm{NE}(\Lambda)\}. \tag{20}$$

In the case of $m = 1$, this definition coincides with (18).

The introduced notion can be used for an alternative definition of the Pareto set. Let $\Theta \subseteq Y$ and $\mathcal{P}(\Theta) = \Theta$, then

$$\Theta = \mathcal{P}(Y) \Leftrightarrow L(F(\cdot), X, \Theta) = X. \tag{21}$$

The global ε-optimality criterion is as follows. Let $\Theta \subseteq Y$ and $\mathcal{P}(\Theta) = \Theta$, then

$$\Theta \text{ is the } \varepsilon\text{-Pareto set } \Leftrightarrow L(F(\cdot), X, \Theta - \varepsilon e_m) = X. \tag{22}$$

We consider collection $X_1, \ldots, X_k$, $X_i \subseteq X$ of the subsets of the permissible set and totality of the finite subsets of the set of reachable criterial vectors $\Lambda_1, \ldots, \Lambda_k$, $\Lambda_i \subseteq Y$. Let $\mu_i(\cdot)$ be the minorant for the vector function $F(\cdot)$ on the set X_i, that is, $\mu_i(x) \leqslant F(x)$ for each $x \in X_i$. Let given be the totality of subsets $\mathcal{L}_1, \ldots, \mathcal{L}_k$ of the set X satisfying the following relations for the given $X_i, \Lambda_i, \mu_i(\cdot)$:

$$\mathcal{L}_i \subseteq \mathcal{L}(\mu_i(\cdot), X_i, \Lambda_i - \varepsilon \times e_m), \quad i = 1, \ldots, k, \tag{23}$$

where $\Lambda_i - \varepsilon e_m = \{x : x = \lambda - \varepsilon \times e_m \text{ for some } \lambda \in \Lambda_i\}$.

We say that the totality of the sets $\{\mathcal{L}_i\}$ *covers the set* X if

$$X = \cup_{i=1}^k \mathcal{L}_i. \tag{24}$$

In this case, the union of the sets $\mathcal{L}_i$ is referred to as *covering*.

Theorem 3. *If the coverage condition* (24) *is satisfied, then the set* $Y_k = \mathcal{P}\left(\cup_{i=1}^k \Lambda_i\right)$ *is the* ε-*Pareto set for problem* (1).

Proof. We consider an arbitrary point $y_* \in \mathcal{P}(Y)$, $y_* = F(x_*)$, $x_* \in X$. If the coverage condition is satisfied, then $x_* \in \mathcal{L}(\mu_i(\cdot), X_i, \Lambda_i - \varepsilon)$ for some i, $1 \leqslant i \leqslant k$. Consequently, there exists the vector $\lambda_i \in \Lambda_i$ such that $\lambda_i - \varepsilon e_m \leqslant \mu_i(x_*) \leqslant F(x_*)$. Since $Y_k = \mathcal{P}\left(\cup_{i=1}^{k} \Lambda_i\right)$, there is $y_\varepsilon \in Y_k$ such that $y_\varepsilon \leqslant \lambda_i$. Whence it follows that $y_\varepsilon - \varepsilon e_m \leqslant \lambda_i - \varepsilon \times e_m \leqslant F(x_*) = y_*$. Validity of condition (5) is established in virtue of arbitrariness of selecting $y_* \in \mathcal{P}(Y)$. Condition (6) follows from (3).

The method of non-uniform coverage constructs the set Y_k and the covering set $\{\mathcal{L}_i\}$ satisfying the conditions of Theorem 3. These means will be discussed in detail in what follows.

4.2. Determination of the Set $\mathcal{L}_i$

Determination of the nonempty set $\mathcal{L}_i$ immediately from definition (23) is a challenge that can be simplified using the evident relation

$$\mathcal{L}(\mu_i(\cdot), X_i, \Lambda_i - \varepsilon e_m) = \cup_{\lambda \in \Lambda_i} \mathcal{L}(\mu_i(\cdot), X_i, \lambda - \varepsilon e_m), \tag{25}$$

where $\mathcal{L}(\mu_i(\cdot), X_i, \lambda - \varepsilon e_m) = \{x \in X_i : \mu_i(x) \geqslant \lambda - \varepsilon e_m\}$. It follows from (25) that the set $\mathcal{L}_i$ can be sought in the form of a union

$$\mathcal{L}_i = \cup_{\lambda \in \Lambda_i} \mathcal{L}_i^\lambda, \tag{26}$$

where $\mathcal{L}_i^\lambda \subseteq \mathcal{L}(\mu_i(\cdot), X_i, \lambda - \varepsilon e_m)$. The sets $\mathcal{L}_i^\lambda$ are found easier than $\mathcal{L}_i$. We indicate two methods for determination of the set $\mathcal{L}_i^\lambda$.

1. Let $\mu_i(\cdot)$ be an arbitrary minorant for the vector function $F(\cdot)$ on the set X_i. Let also the means for determination of the minimum $\alpha_i^{(j)}$ of the function $\mu_i^{(j)}(\cdot)$ on the set X_i, $\alpha_i = (\alpha_i^{(1)}, \ldots, \alpha_i^{(m)})$ be known for each $j \in M$. We assume that

$$\mathcal{L}_i^\lambda = \begin{cases} X_i & \text{if } \alpha_i \geqslant \lambda - \varepsilon e_m \\ \emptyset, & \text{otherwise.} \end{cases}$$

Then, according to (26)

$$\mathcal{L}_i = \begin{cases} X_i & \text{if there is } \lambda \in \Lambda_i \text{ such that } \alpha_i \geqslant \lambda - \varepsilon e_m \\ \emptyset, & \text{otherwise.} \end{cases} \tag{27}$$

This construction of the set $\mathcal{L}_i$ can be conveniently used in the case where the minimum of the minorant $\mu_i^{(j)}$ is readily determined on the set X_i.

2. Let $\mu_i(\cdot)$ be an arbitrary minorant for the vector function $F(\cdot)$ on the set X_i and the means of determining the set $K_i^j \subseteq \mathcal{L}(\mu_i^{(j)}(\cdot), X_i, \lambda^{(j)} - \varepsilon)$ be known for each index j, $1 \leqslant j \leqslant m$. Then, we assume that

$$\mathcal{L}_i^\lambda = \cap_{j=1}^{m} K_i^j. \tag{28}$$

Two following minorants will be used in what follows. If the function $f^{(j)}(x)$ satisfies the Lipschitz condition with the constant l_i^j on the set X_i, then according to [11] the minorant for $f(x)$ is given by the function

$$\mu_i^{(j)}(x) = f^{(j)}(c_i) - l_i^j \|x - c_i\|, \tag{29}$$

where $c_i \in X_i$. Only this case was considered in [8].

If the gradient $f_x^{(j)}(\cdot)$ of the differentiated function $f^{(j)}(\cdot)$ meets the Lipschitz condition with the constant l_i^j, then the minorant

$$\mu_i^{(j)}(x) = f^{(j)}(c_i) + \langle f_x^{(j)}(c_i), x - c_i \rangle - \frac{l_i^j}{2}\|x - c_i\|^2 \tag{30}$$

is used [17].

The values of the minorants (29) and (30) at the point c_i coincide with the value of the criterion $f^{(j)}(\cdot)$ minorated at this point. That is why these minorants are referred to as the *reference minorants*, and the point c_i, as the *reference point*.

In practice, the set X_i usually is an n-dimensional parallelepiped. There are analytical formulas for determination of the minimum of the minorants introduced on the n-dimensional parallelepiped. Therefore, no problems arise at using rule (27) to determine the set $\mathcal{L}_i$.

For the minorant (29), the set $\mathcal{L}(\mu_i^{(j)}(\cdot), X_i, \lambda^{(j)} - \varepsilon)$ is the intersection of the sphere of radius $\rho_i^j(\lambda) = (f^{(j)}(c_i) - \lambda^{(j)} + \varepsilon)/l_i^j$ centered at the point c_i and the set X_i. The case where a minorant like (29) has the same reference point c_i for all criteria was considered in [8]. At that, the set K_i^j is assumed to be equal to the Lebesgue set, that is, to be defined as the intersection of the sphere $B(c_i, \rho_i^j(\lambda))$ and the set X_i. Then,

$$\mathcal{L}_i^\lambda = B(c_i, \rho_i(\lambda)) \cap X_i,$$

where $\rho_i(\lambda) = \min_{1\leqslant j\leqslant M} \rho_i^j(\lambda)$, and

$$\mathcal{L}_i = B(c_i, \rho_i) \cap X_i, \tag{31}$$

where $\rho_i = \max_{\lambda\in\Lambda_i} \rho_i(\lambda)$.

Therefore,

$$\rho_i = \max_{\lambda\in\Lambda_i} \min_{1\leqslant j\leqslant M} (f^{(j)}(c_i) - \lambda^{(j)} + \varepsilon)/l_i^j.$$

This formula coincides with (3) from [8] to within the notation.

For the minorant (30), the set $\mathcal{L}(\mu_i^{(j)}(\cdot), X_i, \lambda^{(j)} - \varepsilon)$ is the intersection of the set X_i and the sphere of radius

$$\rho_i^j(\lambda) = \sqrt{\frac{2}{l_i^j}\left(\frac{1}{2l_i^j}\|f_x^{(j)}(c_i)\|^2 + f(c_i) - \lambda^{(j)} + \varepsilon\right)}$$

centered at the point $z_i = c_i + f_x^{(j)}/l_i^j$.

In this case, the approach used for the minorant (29) does not work because the spheres have different centers and their intersection is a complicated figure yielding with difficulty to the algorithmic processing. We consider two possibilities of approaching this problem.

The first variant relies on the fact that the set $\mathcal{L}(\mu_i^{(j)}(\cdot), X_i, \lambda^{(j)} - \varepsilon)$ comprises a sphere of radius

$$\rho_i^j(\lambda) = \left(\sqrt{\|f_x^{(j)}(c_i)\|^2 + 2l_i^j(f^{(j)}(c_i) - \lambda^{(j)} - \varepsilon)} - \|f_x^{(j)}(c_i)\|\right)/l_i^j \tag{32}$$

centered at the point c_i. By assuming that the set K_i^j is the intersection of this sphere with the set X_i and reasoning as in the case of minorant (29), we establish that the set $\mathcal{L}_i$ is computable from (31), where $\rho_i^j(\lambda)$ is computed from (32).

In the second variant, the n-dimensional parallelepipeds belonging to the set $\mathcal{L}(\mu_i^{(j)}(\cdot), X_i, \lambda^{(j)} - \varepsilon)$ are taken as the set K_i^j. The method to determine such parallelepiped is described in [20]. The set $\mathcal{L}_i^\lambda$ is the intersection of the parallelepipeds and, consequently, a parallelepiped as well. The set $\mathcal{L}_i$ is a union of parallelepipeds that can be used conveniently in the algorithms.

4.3. Algorithm

We consider a possible realization of the non-uniform coverage method. Construction of the set Y_k is an important part of the proposed algorithm. To construct the set Y_k, in the course of operation of the algorithm supported is a list A of the points of the permissible set X such that the image $F(A)$ of the set A under the map F contains a finite collection of pairwise incomparable points which upon completion of the algorithm is the set Y_k. The values of criteria are stored together with the values of parameters, which allows one to avoid calculation of the criteria at each comparison.

The list A is constructed successively using the **Update** procedure realizing addition of a current point to the set A.

Update procedure (A, x)

Parameters:

A—current list of points.

x—new point.

1. Execute the following actions for each point a from A:
   if $F(a) \leqslant F(x)$, then complete the procedure;
   if $F(x) \leqslant F(a)$, then remove a from A.
2. Add x to A.

The image $F(A)$ of the set A constructed using the given procedure satisfies, obviously, condition (6). The sequence of the added points may be generated by different means which are discussed at length in the description of the entire algorithm.

The methods of local optimization enabling one to determine from the point $x \in X$ the point $x' \in X$ such that $F(x') \leqslant F(x)$ improve accuracy of approximation and accelerate operation of the **Cover** algorithm. This procedure is applied to the points x prior to adding them to the list A (see [12–14]).

Now, we consider the basic algorithm **Cover** which decomposes successively the permissible set X into the subsets, generates and rejects the sets $\mathcal{L}_i$ until they make up the coverage of the permissible set. The list A of permissible points is generated in the course of algorithm's operation.

Cover algorithm

1. Initialize the list of subsets $S = \{X\}$ and the list of points $A = \emptyset$.
2. Take some set X_i from S.
3. Select the point $c_i \in X_i$ and update the list of points A: Update(A, c_i).
4. Determine the set $\mathcal{L}_i$ and its complement $X_i' : X_i' = X_i \setminus \mathcal{L}_i$.
5. If $X_i' \neq \emptyset$, then decompose X_i' into p subsets $\mathcal{Y}_i = \{Y_1^i, \dots, Y_p^i\}$ and add them to S.
6. Remove X_i from S.
7. If S is empty, then complete the algorithm; otherwise, go to step 2.

By using Theorem 3, one can readily demonstrate that after completion of the algorithm the set A is an ε-optimal solution and the set $f(A)$ is the ε-Pareto set for problem (1).

The following variant of the above algorithm was used in the experiments:

- it was assumed that the permissible set is a parallelepiped and all sets X_i are parallelepipeds as well;
- the parallelepiped X_i' is always decomposed into two equal parallelepipeds by the hyperplane passing orthogonally through the middle of the maximum-length edge;
- the center of the parallelepiped X_i is always taken as the point c_i.

4.4. Software Realization

The *Cover* algorithm is software realized on the basis of the *BNB-Solver* packet [18] representing a collection of the C++ classes realizing the general scheme of the branch-and-bound method for the serial and parallel architectures. The program seeking the global extremum is obtained by uniting the functionalities of these classes and the functionality of the problem-dependent modules encapsulating the features of a particular method.

To solve a particular problem, one has to realize the modules computing the objective function $F(x)$, gradients of the components, estimates of the Lipschitz constants and the spectrum boundaries for the components of the objective function. For polynomials of more than one variable, the objects can be calculated mechanically. Therefore, modules were added to support the polynomial objective functions and constraints.

The *Cover* method is readily parallelized. Several flows execute independently the algorithm's iterations 2–7 on the shared-memory multiprocessor systems accessing the common list S. Each flow also supports a local list of subsets to prevent losses at synchronization of access to the general list. Part of the subsets from this list is copied periodically to the general list. As soon as the local list is exhausted, the flow takes from the general list a new subset to be processed. Realization for the distributed-memory systems follows the same lines, but instead of copying the messages are transmitted through the network. Selection of the parameters defining the exchange frequency and the number of data transmitted is pivotal for process control. The issues of parallel realization of the method of non-uniform coverage are discussed in [18].

5. EXPERIMENTAL RESULTS

At comparing the algorithms to seek the minimum of a scalar function, the best one is that which establishes the permissible solution with the least value of the objective function. Comparison of the algorithms of multicriteria optimization is a more complicated problem because here it is impossible to specify here a single parameter for comparison. A detailed review of the procedures for comparison of the approximate solutions of the multicriteria problems can be found in [24]. We use below two numerical indices of those proposed in this book.

The first index is called Hyper Volume (HV) and measures the total volume of the domain made up by the united overlapping parallelepipeds situated between some given (*reference*) point and the points of discrete approximation (Fig. 3). A point r such that $r^{(i)} \geqslant \max_{x \in X} f^{(i)}(x)$, $i = 1, \ldots, m$, is usually taken as the reference. We notice that determination of the reference point needs not to be always a trivial problem. If A and A' are two approximation Pareto sets and $A' \leqslant A$, then the

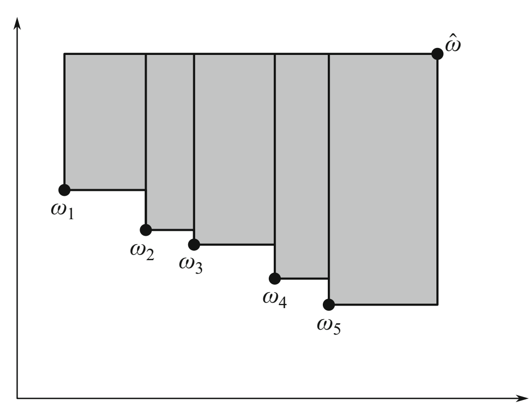

Fig. 3. Hyper Volume (HV) index.

Table 1. Comparison of the results of different algorithms for Example 1

Method	nit	ε	ngen	np	hv
SEMO	500	–	500	221	3.27
MC	500	–	–	22	3.38
NUC	490	0.07	–	36	3.42

HV index of A' exceeds that of A for the same reference point. Stated differently, a greater HV usually corresponds to more accurate approximations.

Beside closeness to the true Pareto set, the quality of the resulting approximation is also characterized by the uniformity of distribution of its points. For the same number of points, that approximation is preferable where the points are distributed more uniformly. The following index (33) allows one to make a rough estimate of this distribution. For the approximation A with k points, we determine

$$\mathrm{UD}(A) = \sqrt{\sum_{i=1}^{k}(d_i - d)^2}, \qquad d = \left(\sum_{i=1}^{k} d_i\right)/k, \tag{33}$$

where d_i is the minimal distance from the point numbered i to the rest of the points A. The smaller values of this index usually correspond to more uniform distributions of points. In particular, if all minimal distances are identical, the index takes on zero value. We notice that index (33) is not meaningful for problems where the Pareto set is disconnected.

Three algorithms were compared in the experiments:

1) NUC: non-uniform coverage method;
2) MC: Monte Carlo method, a stochastic algorithm where the random points are distributed uniformly in the permissible domain and the discrete approximation is constructed using he **Update** procedure;
3) SEMO: genetic algorithm SEMO from the PISA library [25].

The following parameter values were compared:

1) nit—the number of calls of functions computing the criteria;
2) ε—accuracy, this parameter is meaningful only for the method of non-uniform coverage and defines the value ε of the resulting ε-Pareto set;
3) ngen—number of generations, this parameter is meaningful only for the genetic algorithm;
4) np—the number of points in the discrete approximation;
5) hv—value of the Hyper Volume index;
6) ud–value of the index of uniformity of distribution of points in the approximation.

We consider two examples for comparison.

Example 1. Consider problem (1) where the criteria are given by

$$f^{(1)}(x^{(1)}, x^{(2)}) = x^{(1)}, \quad f^{(2)}(x^{(1)}, x^{(2)}) = \min(|x^{(1)} - 1|,\ 1.5 - x^{(1)}) + x^{(2)} + 1,$$

and the permissible domain obeys the inequalities $0 \leqslant x^{(1)} \leqslant 2$ and $0 \leqslant x^{(2)} \leqslant 2$. In this example, the boundary of the Pareto set consists of two segments. The first segment connects the points $(0,\ 2)$ and $(1,\ 1)$, and the second, the points $(1.5,\ 1)$ and $(2,\ 0.5)$.

The parameters of the three algorithms were selected so as to have approximately the number of calculations of criteria close to 500 for them all. Table 1 compiles the experimental results of different algorithms. Dash in the table means that for the given method the parameter is meaningless. The uniformly index was not calculated because the Pareto set is disconnected.

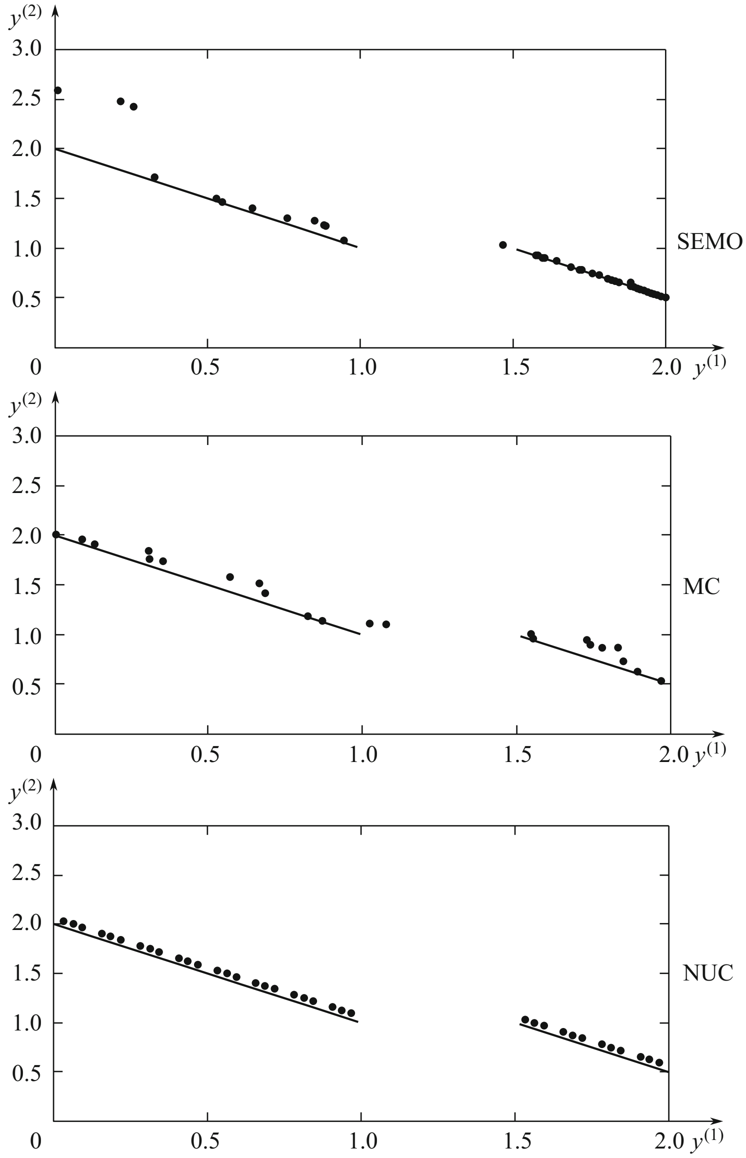

Fig. 4. Approximations of the Pareto set obtained for Example 1 using different methods.

Figure 4 depicts discrete approximations of the Pareto set obtained by the SEMO, MC, and NUC methods for Example 1. One can see that the quality of approximation of the method of non-uniform coverages is much superior to the two other methods.

Example 2. In this example, solved is problem (1) with criteria given by

$$f^{(1)}(x) = (x^{(1)} - 1)(x^{(2)})^2 + 1, \quad f^{(2)}(x) = x^{(2)}.$$

The permissible domain obeys the inequalities $0 \leqslant x^{(1)} \leqslant 1$ and $0 \leqslant x^{(2)} \leqslant 1$.

One can readily see that the solution of the given problem in the space of criteria is represented by a segment of the parabola $y^{(1)} = 1 - (y^{(2)})^2$ for $y^{(1)}, y^{(2)} \in [0, 1]$. The problem is nonconvex. As in Example 1, the parameters of the algorithms were selected so as to provide approximately the same number of computations of criteria close to 500. Table 2 compiles the experimental results for different algorithms. Figure 5 depicts the discrete approximations of the Pareto set obtained by the methods SEMO, MC, and NUC for Example 2.

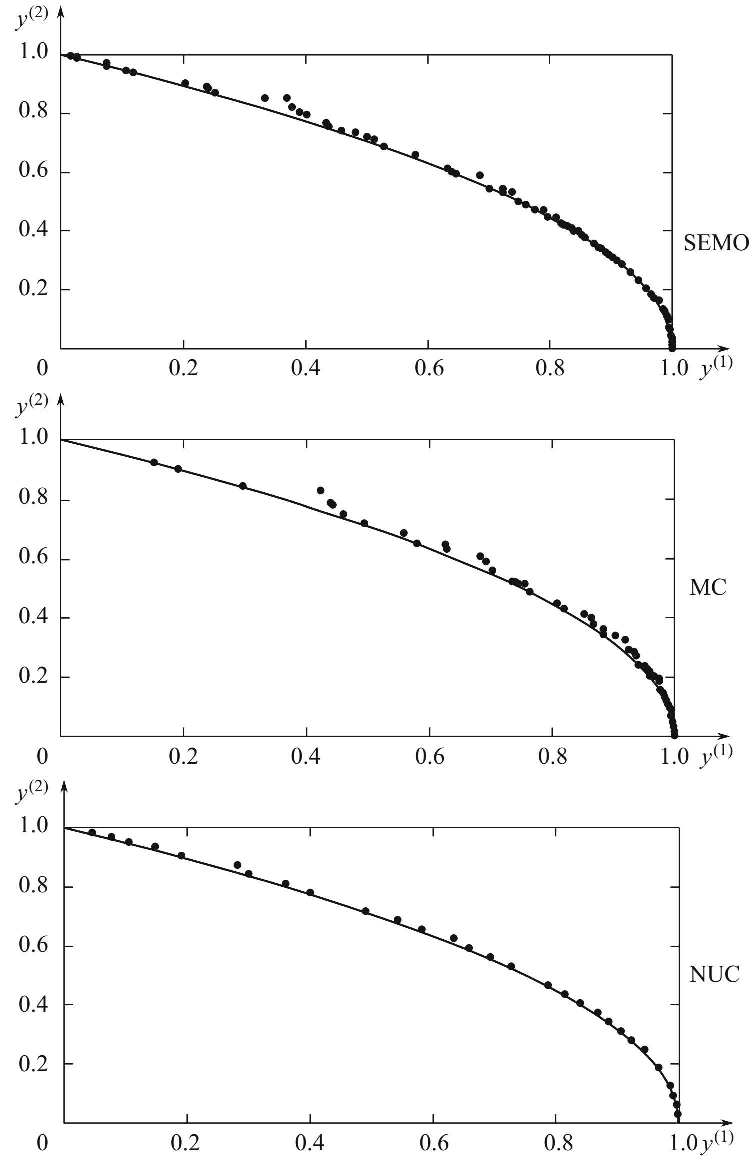

Fig. 5. Approximations of the Pareto set obtained by different methods for Example 2.

Analysis of Table 2 and the graphs in Fig. 5 shows that in terms of the Hyper Volume index the method of non-uniform coverages is much superior to the stochastic algorithm and is somewhat inferior to the genetic algorithm. At that, the genetic algorithm constructed an approximation with the three-fold number of points as compared with the method of non-uniform coverages. The ud index characterizing uniformity of point distribution and analysis of the graphs show that the points constructed by the method of non-uniform coverages have a much more uniform distribution.

Table 2. Comparison of the results of different algorithms for Example 2

Method	nit	ε	ngen	np	hv	ud
SEMO	500	–	500	104	0.312	1.116
MC	500	–	–	67	0.300	1.277
NUC	515	0.0675	–	29	0.306	0.210

It deserves noting that in distinction to the genetic and stochastic algorithms, the method of non-uniform coverages guarantees the ε-optimality for the given ε.

6. CONCLUSIONS

The present paper proposed a new approach to approximation of the Pareto set on the basis of the method of non-uniform coverages. It features two basic advantages. First, it enables one to construct the ε-Pareto set for the given ε, that is, guarantees the ε-optimality of the approximation. This feature is unique and not found in other approaches to the problem of multicriteria optimization.

Second, it follows from the experiments that the points of discrete approximation obtained by the method of non-uniform coverages are distributed more uniformly as compared with the approximations by the other discussed methods. For the same cardinality of the approximation set, the method of non-uniform coverages provides, therefore, better approximations, which is its unconditional merit from the practical point of view.

ACKNOWLEDGMENTS

This work was supported by the Russian Foundation for Basic Research, projects nos. 11-01-12136-ofi-m, 10-07-00700-a and 10-07-00640-a. The authors are indebted to A.V. Lotov for the attention paid to the work and valuable recommendations for its improvement.

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This paper was recommended for publication by A.A. Lazarev, a member of the Editorial Board