Graphical approach for solving combinatorial problems.

Аннотация

In this paper Graphical approach based on the dynamical programming method are presented. We have constructed two algorithms for the knapsack and partition problem based on this approach. Algorithms have been compared .... (Проведен сравнительный анализ предлагаемого метода с известными алгоритмами решения этих задач.)

Introduction

The following combinatorial problems are considered:
Integer Partitioning. Given a set of n positive integer points

$$B = \{b_1, b_2, \ldots, b_n\}.$$

The problem is to find a partition of B into B_1 and B_2, to minimize

$$|\sum_{b_i \in B_1} b_i - \sum_{b_i \in B_2} b_i| \longrightarrow \min . \qquad (1)$$

Knapsack problem.

$$\begin{cases} f(x) = \sum_{i:=1}^{n} c_i x_i \longrightarrow \max \\ \sum_{i:=1}^{n} a_i x_i \leq A, \\ x_i \in \{0,1\},\ i = 1, \ldots, n. \end{cases} \qquad (2)$$

When $c_i = a_i = b_i, i = 1, 2, \ldots, n$,and $A = \frac{1}{2}\sum_{j=1}^{n} b_j$, problems (1) and (2) are identical.

1 Graphical algorithm for the Integer Partitioning (Partition) problem

In Graphical algorithm on steps $i = 1, 2, \ldots, n$ the following function are constructed:

$$F_i^1(t) = |\sum_{b_j \in B_1(t-b_i)} b_j - \sum_{b_j \in B_2(t-b_i)} b_j|$$

$$F_i^2(t) = |\sum_{b_j \in B_1(t+b_i)} b_j - \sum_{b_j \in B_2(t+b_i)} b_j|$$

Function $B_1(t)$ (and $B_2(t)$) in each point t from the interval $[-\sum_{j:=1}^n b_j, \ \sum_{j:=1}^n b_j]$ correspondent to a set $\overline{B_1}$. On the step i $B_1(t) \bigcup B_2(t) = \{b_1, b_2, \ldots, b_{i-1}\}, \ \forall \ t$.

1.1 The idea of Graphical algorithm

On the step $j = 1, 2, \ldots, n$ we chosen where put the number b_j, into B_1 or B_2. In each point t we put b_j such that the value $|\sum_{b_j \in B_1} b_j + t - \sum_{b_j \in B_2} b_j|$ is minimized.

The parameter t is the number, that will be added to B_1 on the next steps. If $t < 0$ then $-t$ is the number, that will be added to B_2.

For example, let the number 30 will be added to B_1 and 28 will be added to B_2 then we consider the point $t = 30 - 28 = 2$.

On the each step i from function $F_{i-1}(t) = |\sum_{b_j \in B_1(t)} b_j - \sum_{b_j \in B_2(t)} b_j|$ we construct

$$F_i(t) := \min\{F_{i-1}(t - b_i), F_{i-1}(t + b_i)\} = \min\{F_i^1(t), F_i^2(t)\}$$

$F_0(t) := 0$.

If $F_i^1(t) < F_i^2(t)$ then $B_2(t) := B_2(t - b_i) \bigcup \{b_i\}$ else $B_1(t) := B_1(t + b_i) \bigcup \{b_i\}$.

Piecewise function we can presented like a table:

t_0	t_1	$\ldots$	t_i	t_{i+1}	$\ldots$	t_{m_i}

Tab.1

In interval $[t_i - \frac{t_i - t_{i-1}}{2}, t_i + \frac{t_{i+1} - t_i}{2})$ we have $F_i(t) = |t - t_i|$, i.e. in $t = t_i$ the equation $F_i(t) = 0$ holds.

Each interval $[t_i - \frac{t_i - t_{i-1}}{2}, t_i + \frac{t_{i+1} - t_i}{2})$ correspondent to some partition $(\overline{B}_1; \overline{B}_2)$, i.e. $B_1(t^1) = B_1(t^2) = \overline{B}_1, \ \forall t^1, t^2 \in [t_i - \frac{t_i - t_{i-1}}{2}, t_i + \frac{t_{i+1} - t_i}{2})$ (analogous for $B_2(t)$).

Let on the step $i - 1$ the function $F_{i-1}(t)$ is constructed (see Tab.1).

On the step i we consider $F_{i-1}(t - b_i)$ and $F_{i-1}(t + b_i)$, (see the tables Таб.2 and Таб.3).

$t_0 - b_i$	$t_1 - b_i$	$\ldots$	$t_i - b_i$	$t_{i+1} - b_i$	$\ldots$	$t_{m_i} - b_i$

Таб.2

$t_0 + b_i$	$t_1 + b_i$	$\ldots$	$t_i + b_i$	$t_{i+1} + b_i$	$\ldots$	$t_{m_i} + b_i$

Таб.3

We consider intervals $[t^1, t^2]$, $t^1, t^2 \in \{t_0 - b_i, t_1 - b_i, \ldots, t_{m_i} - b_i, t_0 + b_i, t_1 + b_i, \ldots, t_{m_i} + b_i\}$, where function $F_{i-1}(t - b_i)$ (and $F_{i-1}(t + b_i)$) depends from one equation.

Obviously, we have the table for function $F_i(t)$ consists only points $\{t_0 - b_i, t_1 - b_i, \ldots, t_{m_i} - b_i, t_0 + b_i, t_1 + b_i, \ldots, t_{m_i} + b_i\}$.

We consider points t^1 and t^2 if $[t^1, t^2] \in [-\sum_{j:=1}^{n} b_j,\ \sum_{j:=1}^{n} b_j]$. So the number of points is less or equal $2 * m_i$.

We have the optimal solution $(B_1(0), B_2(0))$ on the step n and the optimal value $F_n(0)$.

1.2 Cutting of a considered interval

On the step n we consider only point $t = 0$. That's why on the step $n - 1$ we compute the solution (function $F_{n-1}(t)$) only in points $t \in [-b_n, b_n]$. Analogous, on the step $n - 2$ we consider the interval $[-b_n - b_{n-1}, b_n + b_{n-1}]$.

So on each step i we look interval $[-\sum_{j:=i+1}^{n} b_j, \sum_{j:=i+1}^{n} b_j]$ instead of $[-\sum_{j:=1}^{n} b_j,\ \sum_{j:=1}^{n} b_j]$.

Therefore we suggest to enumerate b_j according to $b_1 \geq \ldots \geq b_n$.

1.3 An Instance.

$B = \{100, 70, 50, 20\}$.

$F(0)(t) := 0,\ B_1(t) = \emptyset, B_2(t) = \emptyset, \forall t$

Step 1. $b_1 = 100$. We consider two points $0 + 100$ и $0 - 100$ and three intervals $[-240, -100], [-100, 100], [100, 240]$ (according to "Cutting" : $[-140, -100], [-100, 100], [100, 140]$)

On the interval $[-240, 0]$ we have $B_1(t) = \{100\}, B_2(t) = \emptyset$, and for $t \in [0, 240]$ the optimal partition is $B_1(t) = \emptyset, B_2(t) = \{100\}$.

The computational results and function $F_1(t)$ (in "full" view) are presented on the Fig.1.

We save the following table:

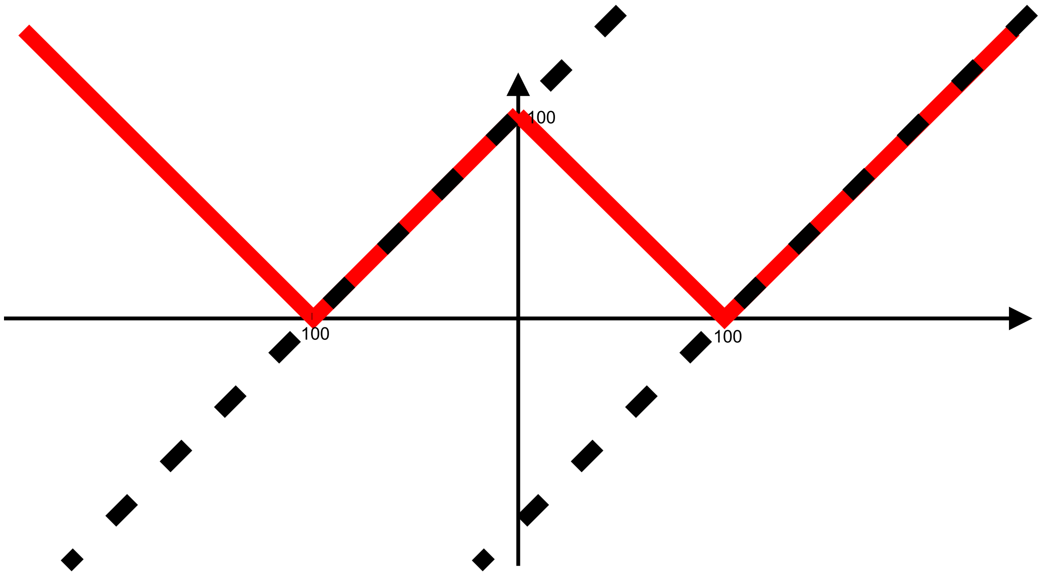

Рис. 1:

-100	100
$(100;)$	$(;100)$

Step 2. $b_2 = 70$. We consider the following points: $-100 - 70$, $-100 + 70 = -30$, $100 - 70 = 30$, $100 + 70 = 170$ and five intervals $[-240, -170], [-170, -30], [-30, 30], [30, 170], [170, 240]$ (or three intervals $[-70, -30], [-30, 30], [30, 70]$ according to "cutting").

On the interval $[-70, 0]$ we have $B_1(t) = \{100\}, B_2(t) = \{70\}$, and for $t \in [0, 70]$ the optimal partition is $B_1(t) = \{70\}, B_2(t) = \{100\}$.

Really we don't consider intervals, but right away we include the points -30 and 30 in the table and compute corresponded partitions. In the point -30 we have partition $B_1(t) = \{100\}, B_2(t) = \{70\}$, and in 30 – $B_1(t) = \{70\}, B_2(t) = \{100\}$.

Functions $F^1(t)$ and $F^2(t)$ are presented on Fig.2.

The computational results and function $F_2(t)$ (in "full" view) are presented on the Fig.3.

We save:

-30	30
$(100;70)$	$(70;100)$

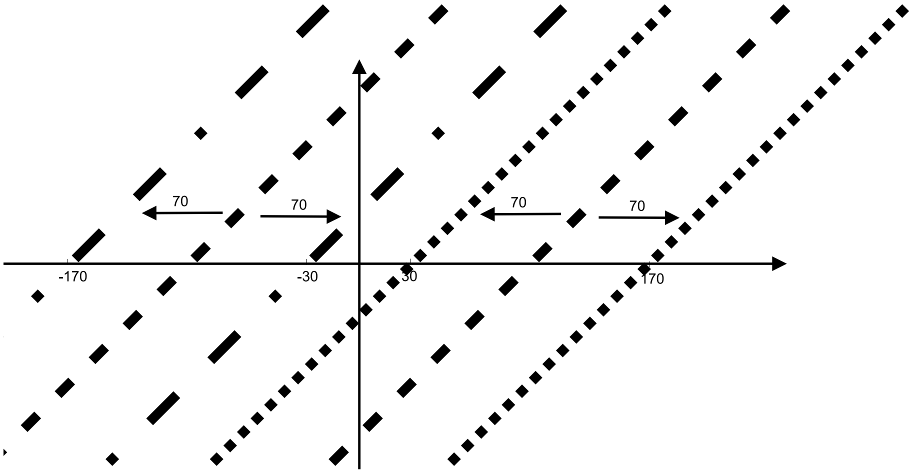

Рис. 2:

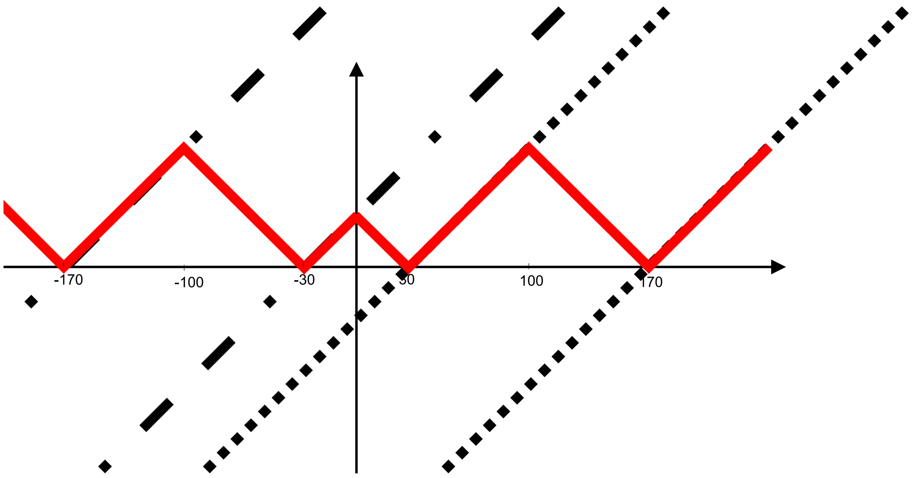

Рис. 3:

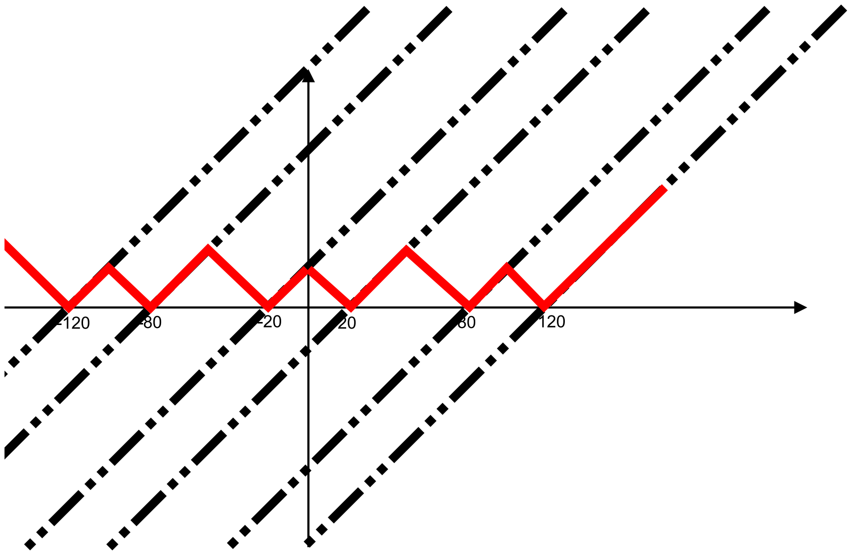

Рис. 4:

Step 3. $b_3 = 50$. 4 points: $-30 - 50 = -80, -30 + 50 = 20, 30 + 50 = 80, 30 - 50 = -20$. We consider interval $[-20, 20]$.

The computational results and function $F_3(t)$ (in "full" view) are presented on the Fig.4.

We save:

-20	20
$(100; 70, 50)$	$(70, 50; 100)$

On the step 4 we have two optimal solutions $B_1(0) = \{100, 20\}, B_2(0) = \{70, 50\}$ and $B_1(0) = \{70, 50\}, B_2(0) = \{100, 20\}$.

So We've considered $2(-100\&100) + 2(-30\&30) + 2(-20\&20) + 1(0) = 7$ points

But in classical Dynamic programming algorithm (with cutting) we need to look $280 + 140 + 40 + 0 = 460$ points.

Function $F_i(t)$ is even. That's why we can to save only half of each table.

1.4 Complexity of computational

Теорема 1 *Graphical algorithm return optimal solution $B_1(0)$ and $B_2(0)$.*

Proof. Let's show that on each step $i = 1, \dots, n$ for each $t \in [-\sum_{j:=\alpha+1}^{n} b_j, \sum_{j:=\alpha+1}^{n} b_j]$ algorithm return an optimal partition $B_1(t) \bigcup B_2(t)$ of set $\{b_1, \dots, b_\alpha\}$.

By mathematic induction:

1. Obviously,on the step $i = 1$ for each $t \in [-\sum_{j:=2}^{n} b_j, \sum_{j:=2}^{n} b_j]$ we have an optimal partition $B_1(t) \bigcup B_2(t)$.

2. Assume, that on the step i for each $t \in [-\sum_{j:=\alpha+1}^{n} b_j, \sum_{j:=\alpha+1}^{n} b_j]$ we have an optimal partition $B_1(t) and B_2(t)$.

3. Let's show that on the step $\alpha+1$ for each $t \in [-\sum_{j:=\alpha+2}^{n} b_j, \sum_{j:=\alpha+2}^{n} b_j]$ algorithm constructs an optimal solution $B_1(t) \bigcup B_2(t)$.

   От противного. Let in the point t algorithm has considered two partition $(B_1(t - b_{i+1});\ B_2(t - b_{i+1}) \bigcup \{b_{i+1}\})$ and $(B_1(t + b_{i+1}) \bigcup \{b_{i+1}\};\ B_2(t + b_{i+1}))$.

   Algorithm chooses partition to minimize $|\sum_{b_j \in B_1} b_j + t - \sum_{b_j \in B_2} b_j|$.

   Let in the point t exists partition $(\overline{B}_1;\ \overline{B}_2)$ such, that

$$|\sum_{b_j \in B_1(t+b_{i+1}) \bigcup \{b_{i+1}\}} b_j + t - \sum_{b_j \in B_2(t+b_{i+1})} b_j| > |\sum_{b_j \in \overline{B}_1} b_j + t - \sum_{b_j \in \overline{B}_2} b_j|$$

   and

$$|\sum_{b_j \in B_1(t-b_{i+1})} b_j + t - \sum_{b_j \in B_2(t-b_{i+1}) \bigcup \{b_{i+1}\}} b_j| > |\sum_{b_j \in \overline{B}_1} b_j + t - \sum_{b_j \in \overline{B}_2} b_j|.$$

   Let $b_{i+1} \in \overline{B}_1$ then we have:

$$|\sum_{b_j \in B_1(t+b_i)} b_j + b_{i+1} + t - \sum_{b_j \in B_2(t+b_i)} b_j| > |\sum_{b_j \in \overline{B}_1 \setminus \{b_{i+1}\}} b_j + b_{i+1} + t - \sum_{b_j \in \overline{B}_2} b_j|,$$

   therefore the partition $(B_1(t+b_\alpha); B_2(t+b_\alpha))$that has been constructed on the step i in the point $t + b_i$ is not optimal because the partition $(\overline{B}_1 \setminus \{b_{\alpha+1}\}; \overline{B}_2)$ is "better". So we have contradiction. Analogous for the case $b_{\alpha+1} \in \overline{B}_2$. $\square$

Algorithm has following properties:

1. There exists the class of integer instances where the number of considered points grow exponential.

2. There exists the class of not-integer instances where the number of considered points grow exponential.

1.5 Experiments

We have compared the effective of graphical algorithm with effective of known algorithm Balsub from [2].

Three groups of experiments have been performed.

1.5.1 Experiment 1

For each integer instance when $n = 4, 5, \dots, 10$ and $40 \geq b_1 \geq b_2 \geq \dots \geq b_n \geq 1$ we have:

1	2	3	4	5	6	7	8	9	10
4	123,410	9	307	328	20	443	640	2	63,684
5	1,086,008	16	444	512	40	564	1000	2	337,077
6	8,145,060	29	542	738	60	687	1440	4	1,140,166
7	53,524,680	48	633	1004	140	811	1960	11	2,799,418
8	314,457,495	76	725	1312	212	933	2560	23	5,348,746
9	1,677,106,640	115	814	1660	376	1053	3240	83	8,488,253
10	8,217,822,536	168	905	2050	500	1172	4000	416	11,426,171

In the first column – (n); in the second one the number of instances (число сочетаний из $b_{max} + n - 1$ по n, где $b_{max} = 40$); in the third – average (steps); in the fourth – average complexity of Balsub algorithm; in the fifth – average complexity of dynamical programming algorithm; in the sixth – maximal complexity of graphical algorithm ; in the seventh – maximal complexity of Balsub algorithm; in the eighth – maximal complexity of dynamical programming algorithm; in the ninth – the number of instances when the complexity of Balsub less then the complexity of graphical algorithm; in the n tenth – the number of instances when the complexity of Balsub great then the complexity of dynamical programming algorithm.

1.5.2 Experiment 2

For $n = 4, 5, \ldots, 10$ we have constructed 20,000 "initial"instances, when parameters b_i are from uniform distribution in $[1, 200]$. Then for each instance in $n-$dimension space we have decided $1000n$ instances $\{b'_1, \ldots, b'_n\}$ where $b_i - (100+n) \le b'_i \le b_i + (100+n),\ i = 1, 2, \ldots, n$. If we have founded "harder" instance we have begun to consider it. If the "harder"instances hasn't been founded we stop the experiment for this initial instance.

We have:

1	2	3	4	5	6	7	8	9
4	8	1463	1591	16	2196	3080	4	2
5	16	2191	2490	40	2797	44675	8	3
6	29	2700	3586	60	3401	6570	12	4
7	50	3145	4881	140	4006	8729	15	6
8	87	3600	6362	216	4617	11056	19	8
9	149	4050	8059	464	5241	13644	52	21
10	245	4499	9930	656	5815	16730	24	10

10	11	12	13	14	15	16	17	18
4	6	1310	1604	20	2207	3200	10504	8970
5	17	2482	3102	40	2811	5000	6641	3642
6	26	2884	3932	60	3418	7176	3000	3170
7	54	3353	6794	136	4029	9800	1101	88
8	82	3849	7039	220	4645	12112	333	377
9	144	4109	11803	476	5232	16200	86	1
10	240	4732	12410	676	5854	18840	18	3

In the first and tenth columns – (n); in the columns 2nd, 3rd, 4th – average complexity of algorithms: graphical, Balsub and dynamical programming algorithm in the "initial point"; in the columns 5th, 6th, 7th – maximal complexity of algorithms in the "initial point"

in 8th and 9th – maximal and average numbers of change of instance; in the columns 11th-13th – average complexity of algorithms: graphical, Balsub and dynamical programming algorithm in the "final point"; in the columns 14th, 15th, 16th – maximal complexity of algorithms in the "final point"in the 17th and 18th – the number of instances when the complexity of Balsub great then the complexity of dynamical programming algorithm for the "initial"and for the "final"points.

For all instances ("final"and "initial") the complexity time of Balsub are great than the complexity of graphical algorithm except 38 instances for $n = 4$ and 2 instances for $n = 5$ from 20000 "final" instances.

1.5.3 Experiment 3

We consider the instances for $n = 4, 5, \ldots, 10$. For each n we find solution for EACH instance, where

$$30 \geq b_1 \geq b_2 \geq \ldots \geq b_n.$$

For each instances Graphical algorithm has been used and then the number of considered points has been calculated.

Define $NP_j,\ j = 1, \ldots, n$ – the number of considered points on the step j. We group instances by value $\sum_{j:=1}^{n} NP_j$.

For example, the rows

1	2	3	4	5	6	7	8
(30,29,29,29)	(1,1,1,0)	3	1	29	15	12767	32%

indicate:

12767 instances have been considered where $\sum_{j:=1}^{n} NP_j = 3$.
1st row: the last considered instance with such value $\sum_{j:=1}^{n} NP_j$.
2nd row: $(NP_1, NP_2, NP_3, NP_4) = (1, 1, 1, 0)$
3rd row: $\sum_{j:=1}^{n} NP_j = 3$.
4th row: $NP_{max} = 1$.
5th row: $b_1 - NP_{max} = 30 - 1 = 29$.
6th row: $n^2 - NP_{max} = 16 - 1 = 15$.

n=4

1	2	3	4	5	6	7	8
(30,26,2,1)	(0,0,0,0)	0	0	30	16	5148	13%
(30,30,30,29)	(1,1,0,0)	2	1	29	15	19156	47%
(30,29,29,29)	(1,1,1,0)	3	1	29	15	12767	32%
(30,30,29,29)	(1,1,1,1)	4	1	29	15	3294	8%
(30,30,30,30)	(1,2,1,1)	5	2	28	14	27	0%

n=5

1	2	3	4	5	6	7	8
(30,23,3,2,1)	(0,0,0,0,0)	0	0	30	25	12458	4%
(30,30,30,28,1)	(1,1,0,0,0)	2	1	29	24	76044	24%
(30,28,28,28,1)	(1,1,1,0,0)	3	1	29	24	46236	15%
(30,30,29,29,1)	(1,1,1,1,0)	4	1	29	24	78231	25%
(30,30,30,30,30)	(1,2,1,1,0)	5	2	28	23	33785	11%
(30,30,30,29,1)	(1,2,1,1,1)	6	2	28	23	28107	9%
(30,30,29,29,29)	(1,2,2,2,0)	7	2	28	23	19443	6%
(30,30,29,29,2)	(1,2,2,2,1)	8	2	28	23	6022	2%
(30,29,28,28,3)	(1,2,3,2,1)	9	3	27	22	10378	3%
(30,29,28,27,4)	(1,2,3,3,1)	10	3	27	22	1960	1%

n=6

1	2	3	4	5	6	7	8
(30,19,4,3,2,1)	(0,0,0,0,0,0)	0	0	30	36	18275	1%
(30,30,30,26,2,1)	(1,1,0,0,0,0)	2	1	29	35	168551	9%
(30,26,26,26,2,1)	(1,1,1,0,0,0)	3	1	29	35	79669	4%
(30,30,28,28,2,1)	(1,1,1,1,0,0)	4	1	29	35	188635	10%
(30,30,30,30,30,29)	(1,2,1,1,0,0)	5	2	28	34	242473	13%
(30,30,30,29,29,29)	(1,2,1,1,1,0)	6	2	28	34	207001	11%
(30,30,30,30,29,29)	(1,2,1,1,1,1)	7	2	28	34	209921	11%
(30,30,29,29,29,28)	(1,2,2,2,1,0)	8	2	28	34	211024	11%
(30,30,30,30,30,30)	(1,2,2,2,1,1)	9	2	28	34	205619	11%
(30,30,29,29,28,28)	(1,2,2,2,2,1)	10	2	28	34	122324	6%
(30,30,29,29,29,29)	(1,2,2,3,2,1)	11	3	27	33	107310	6%
(30,29,28,28,28,27)	(1,2,3,3,2,1)	12	3	27	33	81995	4%
(30,29,28,27,26,26)	(1,2,3,3,3,1)	13	3	27	33	32774	2%
(30,29,28,27,27,27)	(1,2,3,4,3,1)	14	4	26	32	6028	0%
(30,27,26,25,24,24)	(1,2,3,4,4,1)	15	4	26	32	1865	0%

n=7

1	2	3	4	5	6	7	8
(30,14,5,4,3,2,1)	(0,0,0,0,0,0,0)	0	0	30	49	19985	0%
(30,30,30,23,3,2,1)	(1,1,0,0,0,0,0)	2	1	29	48	256213	3%
(30,23,23,23,3,2,1)	(1,1,1,0,0,0,0)	3	1	29	48	92737	1%
(30,30,26,26,4,2,1)	(1,1,1,1,0,0,0)	4	1	29	48	250392	3%
(30,30,30,30,30,28,1)	(1,2,1,1,0,0,0)	5	2	28	47	441932	4%
(30,30,30,28,28,28,1)	(1,2,1,1,1,0,0)	6	2	28	47	569612	6%
(30,30,30,30,29,29,1)	(1,2,1,1,1,1,0)	7	2	28	47	698585	7%
(30,30,30,30,29,28,1)	(1,2,1,1,1,1,1)	8	2	28	47	644642	7%
(30,30,30,30,30,30,30)	(1,2,2,2,1,1,0)	9	2	28	47	932610	9%
(30,30,30,30,30,29,1)	(1,2,2,2,1,1,1)	10	2	28	47	864670	9%
(30,30,30,30,29,29,29)	(1,2,2,2,2,2,0)	11	2	28	47	909580	9%
(30,30,30,30,29,29,2)	(1,2,2,2,2,2,1)	12	2	28	47	822596	8%
(30,30,30,29,29,29,29)	(1,2,2,3,3,2,0)	13	3	27	46	731817	7%
(30,30,30,29,29,29,3)	(1,2,2,3,3,2,1)	14	3	27	46	649333	7%
(30,30,30,29,29,28,4)	(1,2,2,3,3,3,1)	15	3	27	46	522244	5%
(30,30,30,29,28,28,5)	(1,2,2,3,4,3,1)	16	4	26	45	398262	4%
(30,30,30,29,28,27,6)	(1,2,2,3,4,4,1)	17	4	26	45	287696	3%
(30,30,29,29,28,28,4)	(1,2,3,4,4,3,1)	18	4	26	45	210351	2%
(30,30,29,29,28,26,6)	(1,2,3,4,4,4,1)	19	4	26	45	160513	2%
(30,30,29,29,27,27,6)	(1,2,3,4,5,4,1)	20	5	25	44	118452	1%
(30,30,29,29,27,25,8)	(1,2,3,4,5,5,1)	21	5	25	44	83272	1%
(30,30,29,28,28,25,8)	(1,2,3,5,5,5,1)	22	5	25	44	54024	1%
(30,30,29,28,27,26,8)	(1,2,3,5,6,5,1)	23	6	24	43	39255	0%
(30,30,29,28,27,24,10)	(1,2,3,5,6,6,1)	24	6	24	43	32535	0%
(30,30,29,28,26,25,10)	(1,2,3,5,7,6,1)	25	7	23	42	24183	0%
(30,30,29,28,26,23,12)	(1,2,3,5,7,7,1)	26	7	23	42	12353	0%
(30,29,28,28,25,22,12)	(1,2,4,5,7,7,1)	27	7	23	42	9858	0%
(30,29,28,27,26,24,10)	(1,2,4,7,7,6,1)	28	7	23	42	6185	0%
(30,29,28,27,26,22,12)	(1,2,4,7,7,7,1)	29	7	23	42	4207	0%
(30,29,28,27,25,23,12)	(1,2,4,7,8,7,1)	30	8	22	41	2764	0%
(30,29,28,27,25,21,14)	(1,2,4,7,8,8,1)	31	8	22	41	1965	0%
(30,29,28,27,24,22,14)	(1,2,4,7,9,8,1)	32	9	21	40	1019	0%
(30,29,28,27,24,20,16)	(1,2,4,7,9,9,1)	33	9	21	40	256	0%
(30,29,28,26,23,20,18)	(1,2,4,7,10,9,1)	34	10	20	39	54	0%

n=8

1	2	3	4	5	6	7	8
(30,8,6,5,4,3,2,1)	(0,0,0,0,0,0,0,0)	0	0	30	64	19994	0%
(30,30,30,19,4,3,2,1)	(1,1,0,0,0,0,0,0)	2	1	29	63	298576	1%
(30,19,19,19,4,3,2,1)	(1,1,1,0,0,0,0,0)	3	1	29	63	93620	0%
(30,30,23,23,7,3,2,1)	(1,1,1,1,0,0,0,0)	4	1	29	63	264898	1%
(30,30,30,30,30,26,2,1)	(1,2,1,1,0,0,0,0)	5	2	28	62	566538	1%
(30,30,30,26,26,26,2,1)	(1,2,1,1,1,0,0,0)	6	2	28	62	758764	2%
(30,30,30,30,28,28,2,1)	(1,2,1,1,1,1,0,0)	7	2	28	62	1192753	3%
(30,30,30,30,29,27,2,1)	(1,2,1,1,1,1,1,0)	8	2	28	62	1163389	3%
(30,30,30,30,30,30,30,29)	(1,2,2,2,1,1,0,0)	9	2	28	62	1632613	4%
(30,30,30,30,30,29,29,29)	(1,2,2,2,1,1,1,0)	10	2	28	62	1784390	4%
(30,30,30,30,30,30,29,29)	(1,2,2,2,1,1,1,1)	11	2	28	62	2254750	5%
(30,30,30,30,29,29,29,28)	(1,2,2,2,2,2,1,0)	12	2	28	62	2519292	5%
(30,30,30,30,29,29,29,27)	(1,2,2,2,2,2,1,1)	13	2	28	62	2649418	6%
(30,30,30,30,30,30,30,30)	(1,2,2,3,2,2,1,1)	14	3	27	61	2771927	6%
(30,30,30,30,29,29,29,29)	(1,2,2,2,2,3,2,1)	15	3	27	61	2849295	6%
(30,30,30,29,29,29,29,29)	(1,2,2,3,3,3,2,0)	16	3	27	61	2762838	6%
(30,30,30,29,29,29,29,28)	(1,2,2,3,3,3,2,1)	17	3	27	61	2660861	6%
(30,30,30,29,29,28,27,27)	(1,2,2,3,3,3,3,1)	18	3	27	61	2546252	6%
(30,30,30,29,29,28,28,28)	(1,2,2,3,3,4,3,1)	19	4	26	60	2329907	5%
(30,30,30,29,28,28,28,27)	(1,2,2,3,4,4,3,1)	20	4	26	60	2077863	5%
(30,30,30,29,28,27,26,26)	(1,2,2,3,4,4,4,1)	21	4	26	60	1854179	4%
(30,30,30,29,28,27,27,27)	(1,2,2,3,4,5,4,1)	22	5	25	59	1653666	4%
(30,30,30,27,26,25,24,24)	(1,2,2,3,4,5,5,1)	23	5	25	59	1461800	3%
(30,30,29,29,28,26,26,26)	(1,2,3,4,4,5,4,1)	24	5	25	59	1292207	3%
(30,30,29,29,27,27,26,26)	(1,2,3,4,5,5,4,1)	25	5	25	59	1126123	2%
(30,30,29,29,27,27,27,27)	(1,2,3,4,5,6,4,1)	26	6	24	58	962855	2%
(30,30,29,29,27,25,25,25)	(1,2,3,4,5,6,5,1)	27	6	24	58	842650	2%
(30,30,29,28,28,25,25,25)	(1,2,3,5,5,6,5,1)	28	6	24	58	725658	2%
(30,30,29,28,27,26,25,25)	(1,2,3,5,6,6,5,1)	29	6	24	58	618815	1%
(30,30,29,28,27,26,26,26)	(1,2,3,5,6,7,5,1)	30	7	23	57	518376	1%
(30,30,29,28,27,24,24,24)	(1,2,3,5,6,7,6,1)	31	7	23	57	419566	1%
(30,30,29,28,26,25,24,24)	(1,2,3,5,7,7,6,1)	32	7	23	57	335853	1%
(30,30,29,28,26,25,25,25)	(1,2,3,5,7,8,6,1)	33	8	22	56	266554	1%
(30,30,29,28,26,23,23,23)	(1,2,3,5,7,8,7,1)	34	8	22	56	210717	0%
(30,30,29,27,25,23,23,23)	(1,2,3,5,7,9,7,1)	35	9	21	55	169889	0%

1	2	3	4	5	6	7	8
(30,30,29,25,23,21,20,20)	(1,2,3,5,7,9,8,1)	36	9	21	55	137027	0%
(30,30,26,23,22,21,20,20)	(1,2,3,5,7,9,9,1)	37	9	21	55	108779	0%
(30,29,28,27,25,25,25,25)	(1,2,4,7,8,9,6,1)	38	9	21	55	83614	0%
(30,29,28,27,25,23,23,23)	(1,2,4,7,8,9,7,1)	39	9	21	55	60326	0%
(30,29,28,27,25,21,21,21)	(1,2,4,7,8,9,8,1)	40	9	21	55	40353	0%
(30,29,28,27,24,24,24,24)	(1,2,4,7,9,10,7,1)	41	10	20	54	25630	0%
(30,29,28,27,24,22,22,22)	(1,2,4,7,9,10,8,1)	42	10	20	54	13731	0%
(30,29,28,27,24,20,20,20)	(1,2,4,7,9,10,9,1)	43	10	20	54	6741	0%
(30,29,28,26,23,22,22,22)	(1,2,4,7,10,11,8,1)	44	11	19	53	2928	0%
(30,29,28,26,23,20,20,20)	(1,2,4,7,10,11,9,1)	45	11	19	53	1010	0%
(30,29,28,25,22,20,20,20)	(1,2,4,7,10,12,9,1)	46	12	18	52	293	0%
(30,29,28,24,21,18,18,18)	(1,2,4,7,10,13,9,1)	47	13	17	51	83	0%
(30,25,22,20,19,18,18,18)	(1,2,4,7,10,14,9,1)	48	14	16	50	20	0%
(30,23,21,19,18,17,17,17)	(1,2,4,7,11,14,9,1)	49	14	16	50	6	0%
(30,24,21,20,19,18,18,18)	(1,2,4,7,11,14,10,1)	50	14	16	50	1	0%

n=9

1	2	3	4	5	6	7	8
(30,8,6,5,4,3,2,1)	(0,0,0,0,0,0,0,0)	0	0	30	64	19994	0%
(30,30,30,14,5,4,3,2,1)	(1,1,0,0,0,0,0,0,0)	2	1	29	80	304439	0%
(30,19,19,19,4,3,2,1)	(1,1,1,0,0,0,0,0)	3	1	29	63	93620	0%
(30,30,19,19,11,4,3,2,1)	(1,1,1,1,0,0,0,0,0)	4	1	29	80	265781	0%
(30,30,30,30,30,23,3,2,1)	(1,2,1,1,0,0,0,0,0)	5	2	28	79	655359	0%
(30,30,30,23,23,23,3,2,1)	(1,2,1,1,1,0,0,0,0)	6	2	28	79	829990	0%
(30,30,30,30,26,26,4,2,1)	(1,2,1,1,1,1,0,0,0)	7	2	28	79	1398342	1%
(30,30,30,30,28,25,3,2,1)	(1,2,1,1,1,1,1,0,0)	8	2	28	79	1409518	1%
(30,30,30,30,30,30,30,28,1)	(1,2,2,2,1,1,0,0,0)	9	2	28	79	2138241	1%
(30,30,30,30,30,28,28,28,1)	(1,2,2,2,1,1,1,0,0)	10	2	28	79	2330573	1%
(30,30,30,30,30,30,29,29,1)	(1,2,2,2,1,1,1,1,0)	11	2	28	79	3149749	2%
(30,30,30,30,30,30,29,28,1)	(1,2,2,2,1,1,1,1,1)	12	2	28	79	3730845	2%
(30,30,30,30,30,29,29,29,2)	(1,2,2,2,1,2,2,1,0)	13	2	28	79	4335335	2%
(30,30,30,30,30,30,30,30,30)	(1,2,2,3,2,2,1,1,0)	14	3	27	78	5036536	3%
(30,30,30,30,30,30,30,29,1)	(1,2,2,3,2,2,1,1,1)	15	3	27	78	5648246	3%

1	2	3	4	5	6	7	8
(30,30,30,30,30,30,29,29,29)	(1,2,2,3,2,2,2,2,0)	16	3	27	78	6200723	3%
(30,30,30,30,30,30,29,29,2)	(1,2,2,3,2,2,2,2,1)	17	3	27	78	6589419	3%
(30,30,30,30,30,29,29,29,29)	(1,2,2,3,2,3,3,2,0)	18	3	27	78	7246665	4%
(30,30,30,30,30,29,29,29,3)	(1,2,2,3,2,3,3,2,1)	19	3	27	78	7643036	4%
(30,30,30,30,30,29,29,28,4)	(1,2,2,3,2,3,3,3,1)	20	3	27	78	7876191	4%
(30,30,30,30,30,29,28,28,5)	(1,2,2,3,2,3,4,3,1)	21	4	26	77	8065793	4%
(30,30,30,30,30,29,28,27,6)	(1,2,2,3,2,3,4,4,1)	22	4	26	77	8123383	4%
(30,30,30,30,29,29,28,26,6)	(1,2,2,2,3,4,4,4,1)	23	4	26	77	8073016	4%
(30,30,30,30,29,29,29,29,29)	(1,2,2,3,4,5,4,3,0)	24	5	25	76	7965389	4%
(30,30,30,30,29,29,29,29,4)	(1,2,2,3,4,5,4,3,1)	25	5	25	76	7748098	4%
(30,30,30,30,29,29,29,27,6)	(1,2,2,3,4,5,4,4,1)	26	5	25	76	7483335	4%
(30,30,30,30,29,29,28,28,6)	(1,2,2,3,4,5,5,4,1)	27	5	25	76	7233758	4%
(30,30,30,30,29,29,28,26,8)	(1,2,2,3,4,5,5,5,1)	28	5	25	76	6885288	4%
(30,30,30,30,29,29,27,27,8)	(1,2,2,3,4,5,6,5,1)	29	6	24	75	6547791	3%
(30,30,30,30,29,29,27,25,10)	(1,2,2,3,4,5,6,6,1)	30	6	24	75	6210831	3%
(30,30,30,30,29,28,28,25,10)	(1,2,2,3,4,6,6,6,1)	31	6	24	75	5813624	3%
(30,30,30,30,29,28,27,26,10)	(1,2,2,3,4,6,7,6,1)	32	7	23	74	5412947	3%
(30,30,30,30,29,28,27,24,12)	(1,2,2,3,4,6,7,7,1)	33	7	23	74	5026254	3%
(30,30,30,30,29,28,26,25,12)	(1,2,2,3,4,6,8,7,1)	34	8	22	73	4618851	2%
(30,30,30,30,29,28,26,23,14)	(1,2,2,3,4,6,8,8,1)	35	8	22	73	4211323	2%
(30,30,30,29,29,28,27,23,12)	(1,2,2,4,5,7,7,7,1)	36	7	23	74	3831634	2%
(30,30,30,29,29,28,26,24,12)	(1,2,2,4,5,7,8,7,1)	37	8	22	73	3474732	2%
(30,30,30,29,29,28,26,22,14)	(1,2,2,4,5,7,8,8,1)	38	8	22	73	3119842	2%
(30,30,30,29,29,28,25,23,14)	(1,2,2,4,5,7,9,8,1)	39	9	21	72	2788033	1%
(30,30,30,29,29,28,25,21,16)	(1,2,2,4,5,7,9,9,1)	40	9	21	72	2478862	1%
(30,30,30,29,29,27,26,21,16)	(1,2,2,4,5,8,9,9,1)	41	9	21	72	2181683	1%
(30,30,30,29,29,27,25,22,16)	(1,2,2,4,5,8,10,9,1)	42	10	20	71	1910715	1%
(30,30,30,29,29,27,25,20,18)	(1,2,2,4,5,8,10,10,1)	43	10	20	71	1663292	1%
(30,30,30,29,29,27,24,21,18)	(1,2,2,4,5,8,11,10,1)	44	11	19	70	1432074	1%
(30,30,30,29,28,28,25,20,18)	(1,2,2,4,7,8,10,10,1)	45	10	20	71	1226157	1%
(30,30,30,29,28,28,24,21,18)	(1,2,2,4,7,8,11,10,1)	46	11	19	70	1046920	1%
(30,30,30,29,28,27,26,20,18)	(1,2,2,4,7,10,10,10,1)	47	10	20	71	893835	0%
(9,8,7,6,6,6,6,6,6)	(1,2,4,8,10,10,7,4,1)	47	10	-1	71		
(30,30,30,29,28,27,25,21,18)	(1,2,2,4,7,10,11,10,1)	48	11	19	70	758258	0%
(30,30,30,29,28,27,24,22,18)	(1,2,2,4,7,10,12,10,1)	49	12	18	69	639324	0%
(30,30,30,29,28,27,24,20,20)	(1,2,2,4,7,10,12,11,1)	50	12	18	69	533561	0%
(30,30,30,29,28,27,23,21,20)	(1,2,2,4,7,10,13,11,1)	51	13	17	68	444470	0%
(30,30,30,29,28,26,24,21,20)	(1,2,2,4,7,11,13,11,1)	52	13	17	68	363596	0%
(30,30,30,29,28,26,23,22,20)	(1,2,2,4,7,11,14,11,1)	53	14	16	67	295748	0%

1	2	3	4	5	6	7	8
(30,30,30,29,28,26,22,22,21)	(1,2,2,4,7,11,15,11,1)	54	15	15	66	241712	0%
(30,30,29,29,28,26,22,22,20)	(1,2,3,5,7,11,14,11,1)	55	14	16	67	194226	0%
(30,30,29,29,27,25,24,22,20)	(1,2,3,5,8,13,13,10,1)	56	13	17	68	154328	0%
(30,30,29,29,27,25,23,23,20)	(1,2,3,5,8,13,14,10,1)	57	14	16	67	122560	0%
(30,30,29,28,28,26,21,21,21)	(1,2,3,6,8,11,15,11,1)	58	15	15	66	96890	0%
(30,30,29,29,27,25,22,22,22)	(1,2,3,5,8,13,15,11,1)	59	15	15	66	75463	0%
(30,30,29,28,28,25,22,22,20)	(1,2,3,6,8,13,15,11,1)	60	15	15	66	58693	0%
(30,30,29,28,27,26,23,21,20)	(1,2,3,6,10,13,14,11,1)	61	14	16	67	45009	0%
(30,30,29,28,27,26,22,22,20)	(1,2,3,6,10,13,15,11,1)	62	15	15	66	35009	0%
(30,30,29,28,27,25,23,22,20)	(1,2,3,6,10,14,15,11,1)	63	15	15	66	26516	0%
(30,30,29,28,27,25,22,22,21)	(1,2,3,6,10,14,16,11,1)	64	16	14	65	20273	0%
(30,30,29,28,27,24,24,22,20)	(1,2,3,6,10,16,15,11,1)	65	16	14	65	15087	0%
(30,30,29,28,27,24,23,23,20)	(1,2,3,6,10,16,16,11,1)	66	16	14	65	11605	0%
(30,30,29,28,26,24,23,22,22)	(1,2,3,6,11,16,16,11,1)	67	16	14	65	8420	0%
(30,30,29,28,27,24,22,22,22)	(1,2,3,6,10,16,17,12,1)	68	17	13	64	5806	0%
(30,30,29,28,26,23,23,23,23)	(1,2,3,6,11,18,17,11,0)	69	18	12	63	4113	0%
(30,30,29,28,26,23,23,23,22)	(1,2,3,6,11,18,17,11,1)	70	18	12	63	2891	0%
(30,30,27,25,24,23,22,22,21)	(1,2,3,6,11,18,18,11,1)	71	18	12	63	1947	0%
(30,29,28,27,26,23,23,22,20)	(1,2,4,8,12,17,16,11,1)	72	17	13	64	1279	0%
(30,29,28,27,26,23,22,22,21)	(1,2,4,8,12,17,17,11,1)	73	17	13	64	798	0%
(30,29,28,27,25,22,21,21,21)	(1,2,4,7,12,18,18,11,1)	74	18	12	63	458	0%
(30,29,28,27,25,23,22,22,22)	(1,2,4,8,13,18,17,11,1)	75	18	12	63	302	0%
(30,29,28,27,24,22,21,21,20)	(1,2,4,7,13,19,18,11,1)	76	19	11	62	213	0%
(30,29,28,27,26,22,22,22,22)	(1,2,4,8,12,19,18,12,1)	77	19	11	62	80	0%
(30,29,28,27,24,21,21,21,21)	(1,2,4,7,13,20,19,11,1)	78	20	10	61	27	0%
(30,29,28,26,23,20,20,20,20)	(1,2,4,7,13,21,19,11,1)	79	21	9	60	26	0%
(30,29,26,24,23,22,22,22,20)	(1,2,4,8,14,21,18,11,1)	80	21	9	60	16	0%
(30,28,25,23,22,21,21,21,21)	(1,2,4,8,14,21,19,11,1)	81	21	9	60	5	0%
(30,26,23,22,21,20,20,20,20)	(1,2,4,8,15,23,19,11,1)	84	23	7	58	1	0%

n=10

1	2	3	4	5	6	7	8
(30,8,6,5,4,3,2,1)	(0,0,0,0,0,0,0,0)	0	0	30	64	19994	0%
(30,30,30,8,6,5,4,3,2,1)	(1,1,0,0,0,0,0,0,0,0)	2	1	29	99	304449	0%
(30,19,19,19,4,3,2,1)	(1,1,1,0,0,0,0,0)	3	1	29	63	93620	0%
(30,30,19,19,11,4,3,2,1)	(1,1,1,1,0,0,0,0,0)	4	1	29	80	265781	0%
(30,30,30,30,30,19,4,3,2,1)	(1,2,1,1,0,0,0,0,0,0)	5	2	28	98	697722	0%
(30,30,30,19,19,19,4,3,2,1)	(1,2,1,1,1,0,0,0,0,0)	6	2	28	98	851411	0%
(30,30,30,30,23,23,7,3,2,1)	(1,2,1,1,1,1,0,0,0,0)	7	2	28	98	1445055	0%
(30,30,30,30,26,22,4,4,2,1)	(1,2,1,1,1,1,1,0,0,0)	8	2	28	98	1450277	0%
(30,30,30,30,30,30,30,26,2,1)	(1,2,2,2,1,1,0,0,0,0)	9	2	28	98	2339634	0%
(30,30,30,30,30,26,26,26,2,1)	(1,2,2,2,1,1,1,0,0,0)	10	2	28	98	2590785	0%
(30,30,30,30,30,30,28,28,2,1)	(1,2,2,2,1,1,1,1,0,0)	11	2	28	98	3637447	0%
(30,30,30,30,30,30,29,27,2,1)	(1,2,2,2,1,1,1,1,1,0)	12	2	28	98	4396512	1%
(30,30,30,30,30,30,29,26,2,1)	(1,2,2,2,1,1,1,1,1,1)	13	2	28	98	5234322	1%
(30,30,30,30,30,30,30,30,30,29)	(1,2,2,3,2,2,1,1,0,0)	14	3	27	97	6243720	1%
(30,30,30,30,30,30,30,29,29,29)	(1,2,2,3,2,2,1,1,1,0)	15	3	27	97	7370365	1%
(30,30,30,30,30,30,30,30,29,29)	(1,2,2,3,2,2,1,1,1,1)	16	3	27	97	8541233	1%
(30,30,30,30,30,30,29,29,29,28)	(1,2,2,3,2,2,2,2,1,0)	17	3	27	97	9390033	1%
(30,30,30,30,30,30,29,29,29,27)	(1,2,2,3,2,2,2,2,1,1)	18	3	27	97	10811411	1%
(30,30,30,30,30,30,29,29,28,28)	(1,2,2,3,2,2,2,2,2,1)	19	3	27	97	12004423	2%
(30,30,30,30,30,30,30,30,30,30)	(1,2,2,3,3,3,2,2,1,1)	20	3	27	97	13103492	2%
(30,30,30,30,30,29,29,29,29,29)	(1,2,2,3,2,3,3,3,2,0)	21	3	27	97	14368945	2%
(30,30,30,30,30,29,29,29,29,28)	(1,2,2,3,2,3,3,3,2,1)	22	3	27	97	15717390	2%
(30,30,30,30,30,29,29,28,27,27)	(1,2,2,3,2,3,3,3,3,1)	23	3	27	97	16886376	2%
(30,30,30,30,30,29,29,28,28,28)	(1,2,2,3,2,3,3,4,3,1)	24	4	26	96	18013354	2%
(30,30,30,30,30,29,28,28,28,27)	(1,2,2,3,2,3,4,4,3,1)	25	4	26	96	19056476	2%
(30,30,30,30,30,29,28,27,26,26)	(1,2,2,3,2,3,4,4,4,1)	26	4	26	96	20066120	3%
(30,30,30,30,30,29,28,27,27,27)	(1,2,2,3,2,3,4,5,4,1)	27	5	25	95	21081684	3%
(30,30,30,30,30,27,26,25,24,24)	(1,2,2,3,2,3,4,5,5,1)	28	5	25	95	21798133	3%
(30,30,30,30,29,29,29,29,28,28)	(1,2,2,3,4,5,4,4,3,1)	29	5	25	95	22401133	3%
(30,30,30,30,29,29,29,27,26,26)	(1,2,2,3,4,5,4,4,4,1)	30	5	25	95	22961447	3%
(30,30,30,30,29,29,29,29,29,29)	(1,2,2,3,4,5,5,5,3,1)	31	5	25	95	23262787	3%
(30,30,30,30,29,29,28,28,27,27)	(1,2,2,3,4,5,5,5,4,1)	32	5	25	95	23388641	3%
(30,30,30,30,29,29,28,28,28,28)	(1,2,2,3,4,5,5,6,4,1)	33	6	24	94	23397135	3%
(30,30,30,30,29,29,28,26,26,26)	(1,2,2,3,4,5,5,6,5,1)	34	6	24	94	23290605	3%
(30,30,30,30,29,29,27,27,26,26)	(1,2,2,3,4,5,6,6,5,1)	35	6	24	94	23005770	3%
(30,30,30,30,29,29,27,27,27,27)	(1,2,2,3,4,5,6,7,5,1)	36	7	23	93	22658925	3%
(30,30,30,30,29,29,27,25,25,25)	(1,2,2,3,4,5,6,7,6,1)	37	7	23	93	22168453	3%
(30,30,30,30,29,28,28,25,25,25)	(1,2,2,3,4,6,6,7,6,1)	38	7	23	93	21544964	3%
(30,30,30,30,29,28,27,26,25,25)	(1,2,2,3,4,6,7,7,6,1)	39	7	23	93	20838805	3%

1	2	3	4	5	6	7	8
(30,30,30,30,29,28,27,26,26,26)	(1,2,2,3,4,6,7,8,6,1)	40	8	22	92	20101712	3%
(30,30,30,30,29,28,27,24,24,24)	(1,2,2,3,4,6,7,8,7,1)	41	8	22	92	19302438	3%
(30,30,30,30,29,28,26,25,24,24)	(1,2,2,3,4,6,8,8,7,1)	42	8	22	92	18464529	2%
(30,30,30,30,29,28,26,25,25,25)	(1,2,2,3,4,6,8,9,7,1)	43	9	21	91	17600082	2%
(30,30,30,30,29,28,26,23,23,23)	(1,2,2,3,4,6,8,9,8,1)	44	9	21	91	16702173	2%
(30,30,30,30,29,27,25,23,23,23)	(1,2,2,3,4,6,8,10,8,1)	45	10	20	90	15773247	2%
(10,10,9,9,7,5,5,5,5,5)	(1,2,3,4,6,11,8,6,3,1)	45	11	-1	89		
(30,30,30,30,29,25,23,21,20,20)	(1,2,2,3,4,6,8,10,9,1)	46	10	20	90	14840113	2%
(10,10,9,8,8,5,5,5,5,5)	(1,2,3,5,6,11,8,6,3,1)	46	11	-1	89		
(30,30,30,29,29,28,26,22,22,22)	(1,2,2,4,5,7,8,9,8,1)	47	9	21	91	13911032	2%
(10,10,9,8,7,6,6,6,4,4)	(1,2,3,5,8,11,8,5,3,1)	47	11	-1	89		
(30,30,30,29,29,28,25,25,25,25)	(1,2,2,4,5,7,9,10,7,1)	48	10	20	90	12995508	2%
(10,10,9,8,7,6,5,5,5,5)	(1,2,3,5,8,11,8,6,3,1)	48	11	-1	89		
(30,30,30,29,29,28,25,23,23,23)	(1,2,2,4,5,7,9,10,8,1)	49	10	20	90	12118884	2%
(11,11,10,9,8,7,7,7,7,4)	(1,2,3,5,9,12,9,6,2,0)	49	12	-1	88		
(30,30,30,29,29,28,25,21,21,21)	(1,2,2,4,5,7,9,10,9,1)	50	10	20	90	11255001	1%
(12,12,11,10,8,7,7,7,7,4)	(1,2,3,5,9,13,9,6,2,0)	50	13	-1	87		
(30,30,30,29,29,27,26,21,21,21)	(1,2,2,4,5,8,9,10,9,1)	51	10	20	90	10436174	1%
(12,12,11,10,8,7,7,7,7,3)	(1,2,3,5,9,13,9,6,2,1)	51	13	-1	87		
(30,30,30,29,29,27,25,24,24,24)	(1,2,2,4,5,8,10,11,8,1)	52	11	19	89	9649157	1%
(12,12,11,10,8,7,7,7,6,4)	(1,2,3,5,9,13,9,6,3,1)	52	13	-1	87		
(30,30,30,29,29,27,25,22,22,22)	(1,2,2,4,5,8,10,11,9,1)	53	11	19	89	8899970	1%
(12,12,11,10,9,6,6,6,6,6)	(1,2,3,5,7,13,10,7,4,1)	53	13	-1	87		
(30,30,30,29,29,27,25,20,20,20)	(1,2,2,4,5,8,10,11,10,1)	54	11	19	89	8189876	1%
(13,13,10,9,8,7,7,7,7,6)	(1,2,3,5,9,14,10,7,3,0)	54	14	-1	86		
(30,30,30,29,29,27,24,23,23,23)	(1,2,2,4,5,8,11,12,9,1)	55	12	18	88	7516327	1%
(13,13,10,9,8,7,7,7,7,5)	(1,2,3,5,9,14,10,7,3,1)	55	14	-1	86		
(30,30,30,29,29,27,24,21,21,21)	(1,2,2,4,5,8,11,12,10,1)	56	12	18	88	6868675	1%
(13,13,12,11,9,8,8,8,8,5)	(1,2,3,5,10,14,11,7,3,0)	56	14	-1	86		
(30,30,30,29,29,26,23,21,21,21)	(1,2,2,4,5,8,11,13,10,1)	57	13	17	87	6278557	1%
(13,13,12,11,10,7,7,7,7,7)	(1,2,3,5,8,14,11,8,4,1)	57	14	-1	86		
(30,30,30,29,29,25,21,18,18,18)	(1,2,2,4,5,8,11,14,10,1)	58	14	16	86	5714842	1%
(14,14,13,12,10,7,7,7,7,7)	(1,2,3,5,8,15,11,8,4,1)	58	15	-1	85		
(30,30,30,29,28,27,25,23,23,23)	(1,2,2,4,7,10,11,12,9,1)	59	12	18	88	5182024	1%
(14,11,11,9,8,7,7,7,7,7)	(1,2,3,5,9,15,11,8,4,1)	59	15	-1	85		
(30,30,30,29,28,27,25,21,21,21)	(1,2,2,4,7,10,11,12,10,1)	60	12	18	88	4695528	1%

1	2	3	4	5	6	7	8
(14,14,13,12,10,8,8,8,8,8)	(1,2,3,5,10,15,12,8,4,0)	60	15	-1	85		
(30,30,30,29,28,27,24,24,24,24)	(1,2,2,4,7,10,12,13,9,1)	61	13	17	87	4240616	1%
(14,14,13,12,10,8,8,8,8,7)	(1,2,3,5,10,15,12,8,4,1)	61	15	-1	85		
(30,30,30,29,28,27,24,22,22,22)	(1,2,2,4,7,10,12,13,10,1)	62	13	17	87	3820975	0%
(14,14,12,11,10,9,9,9,9,5)	(1,2,3,6,11,15,12,8,3,1)	62	15	-1	85		
(30,30,30,29,28,27,24,20,20,20)	(1,2,2,4,7,10,12,13,11,1)	63	13	17	87	3442759	0%
(15,14,13,12,9,9,9,9,9,4)	(1,2,4,7,13,16,11,7,2,0)	63	16	-1	84		
(30,30,30,29,28,27,23,23,23,23)	(1,2,2,4,7,10,13,14,10,1)	64	14	16	86	3089814	0%
(15,15,14,13,11,8,8,8,8,8)	(1,2,3,5,9,16,13,9,5,1)	64	16	-1	84		
(30,30,30,29,28,27,23,21,21,21)	(1,2,2,4,7,10,13,14,11,1)	65	14	16	86	2768158	0%
(15,15,12,11,10,9,9,9,9,8)	(1,2,3,6,11,16,13,9,4,0)	65	16	-1	84		
(30,30,30,29,28,26,24,21,21,21)	(1,2,2,4,7,11,13,14,11,1)	66	14	16	86	2473073	0%
(16,15,13,11,10,9,9,9,9,6)	(1,2,4,7,12,17,12,8,3,0)	66	17	-1	83		
(30,30,30,29,28,26,23,22,21,21)	(1,2,2,4,7,11,14,14,11,1)	67	14	16	86	2205246	0%
(16,16,13,12,11,10,10,10,10,7)	(1,2,3,6,11,17,14,9,4,0)	67	17	-1	83		
(30,30,30,29,28,26,23,22,22,22)	(1,2,2,4,7,11,14,15,11,1)	68	15	15	85	1964012	0%
(16,16,15,14,12,9,9,9,9,9)	(1,2,3,5,10,17,14,10,5,1)	68	17	-1	83		
(30,30,30,29,28,26,22,22,22,21)	(1,2,2,4,7,11,15,15,11,1)	69	15	15	85	1745005	0%
(16,16,13,12,11,10,10,10,10,9)	(1,2,3,6,11,17,14,10,5,0)	69	17	-1	83		
(30,30,30,29,28,26,22,21,21,21)	(1,2,2,4,7,11,15,16,11,1)	70	16	14	84	1548129	0%
(17,16,15,14,11,9,9,9,9,8)	(1,2,4,7,12,18,13,9,4,0)	70	18	-1	82		
(30,30,30,27,25,24,23,23,23,23)	(1,2,2,4,7,11,16,16,11,1)	71	16	14	84	1372214	0%
(17,17,14,12,11,10,10,10,10,10)	(1,2,3,6,11,18,15,10,5,0)	71	18	-1	82		
(30,30,29,29,28,25,21,20,20,20)	(1,2,3,5,7,11,15,16,11,1)	72	16	14	84	1212748	0%
(17,17,14,12,11,10,10,10,10,9)	(1,2,3,6,11,18,15,10,5,1)	72	18	-1	82		
(30,30,29,29,27,26,21,20,20,20)	(1,2,3,5,8,11,15,16,11,1)	73	16	14	84	1071932	0%
(17,16,15,13,11,10,10,10,10,6)	(1,2,4,7,13,18,14,9,4,1)	73	18	-1	82		
(30,30,29,29,27,25,22,22,21,21)	(1,2,3,5,8,13,15,15,11,1)	74	15	15	85	945624	0%
(18,17,13,12,11,9,9,9,9,9)	(1,2,4,7,11,19,14,10,5,1)	74	19	-1	81		
(30,30,29,29,27,25,22,22,22,22)	(1,2,3,5,8,13,15,16,11,1)	75	16	14	84	830754	0%
(18,17,15,13,12,9,9,9,9,9)	(1,2,4,7,12,19,14,10,5,1)	75	19	-1	81		
(30,30,29,29,27,25,21,21,20,20)	(1,2,3,5,8,13,16,16,11,1)	76	16	14	84	727311	0%
(19,18,15,13,12,11,11,11,11,8)	(1,2,4,7,13,20,15,10,4,0)	76	20	-1	80		
(30,30,29,29,27,25,21,21,21,21)	(1,2,3,5,8,13,16,17,11,1)	77	17	13	83	633672	0%
(19,18,17,15,12,11,11,11,11,8)	(1,2,4,7,14,20,15,10,4,0)	77	20	-1	80		

1	2	3	4	5	6	7	8
(30,30,29,29,27,25,20,18,18,18)	(1,2,3,5,8,13,17,18,10,1)	78	18	12	82	548549	0%
(19,18,17,15,12,11,11,11,11,7)	(1,2,4,7,14,20,15,10,4,1)	78	20	-1	80		
(30,30,29,29,27,25,20,20,20,20)	(1,2,3,5,8,13,17,18,11,1)	79	18	12	82	473240	0%
(19,18,17,15,12,11,11,11,10,8)	(1,2,4,7,14,20,15,10,5,1)	79	20	-1	80		
(30,30,29,28,28,25,20,20,20,20)	(1,2,3,6,8,13,17,18,11,1)	80	18	12	82	407345	0%
(19,18,17,15,12,11,11,11,11,10)	(1,2,4,7,14,20,16,11,5,0)	80	20	-1	80		
(30,30,29,28,27,26,20,18,18,18)	(1,2,3,6,10,13,17,18,10,1)	81	18	12	82	350196	0%
(19,18,17,15,12,11,11,11,11,9)	(1,2,4,7,14,20,16,11,5,1)	81	20	-1	80		
(30,30,29,28,27,26,20,20,20,20)	(1,2,3,6,10,13,17,18,11,1)	82	18	12	82	297239	0%
(20,17,15,13,12,11,11,11,11,10)	(1,2,4,8,14,21,16,11,5,0)	82	21	-1	79		
(30,30,29,28,27,25,21,20,20,20)	(1,2,3,6,10,14,17,18,11,1)	83	18	12	82	251612	0%
(20,17,15,14,13,11,11,11,11,10)	(1,2,4,8,15,21,16,11,5,0)	83	21	-1	79		
(30,30,29,28,27,25,20,19,19,19)	(1,2,3,6,10,14,18,19,10,1)	84	19	11	81	211944	0%
(21,17,14,13,12,11,11,11,11,10)	(1,2,4,8,15,22,16,11,5,0)	84	22	-1	78		
(30,30,29,28,27,25,19,19,19,18)	(1,2,3,6,10,14,19,19,10,1)	85	19	11	81	176197	0%
(21,17,14,13,12,11,11,11,11,9)	(1,2,4,8,15,22,16,11,5,1)	85	22	-1	78		
(30,30,29,28,27,24,22,22,22,22)	(1,2,3,6,10,16,17,18,12,1)	86	18	12	82	146424	0%
(21,17,14,13,12,11,11,11,10,10)	(1,2,4,8,15,22,16,11,6,1)	86	22	-1	78		
(30,30,29,28,27,24,21,21,21,21)	(1,2,3,6,10,16,18,19,11,1)	87	19	11	81	121279	0%
(20,17,15,14,13,11,11,11,11,11)	(1,2,4,8,15,21,17,12,6,1)	87	21	-1	79		
(30,30,29,28,27,24,19,19,18,18)	(1,2,3,6,10,16,20,19,10,1)	88	20	10	80	99935	0%
(21,17,14,13,12,11,11,11,11,11)	(1,2,4,8,15,22,17,12,6,1)	88	22	-1	78		
(30,30,29,28,27,24,20,20,20,20)	(1,2,3,6,10,16,19,20,11,1)	89	20	10	80	82083	0%
(30,30,29,28,26,23,22,22,22,22)	(1,2,3,6,11,18,18,19,11,1)	90	19	11	81	66803	0%
(30,30,29,28,26,23,21,21,20,20)	(1,2,3,6,11,18,19,19,11,1)	91	19	11	81	54316	0%
(20,19,18,16,13,12,12,12,12,12)	(1,2,4,8,15,21,19,13,7,1)	91	21	-1	79		
(30,30,29,28,26,23,21,21,21,21)	(1,2,3,6,11,18,19,20,11,1)	92	20	10	80	43880	0%
(30,30,29,28,26,23,20,20,20,20)	(1,2,3,6,11,18,20,20,11,1)	93	20	10	80	35003	0%
(30,30,29,25,22,20,19,19,18,18)	(1,2,3,6,11,18,23,19,10,1)	94	23	7	77	27279	0%
(30,30,27,24,23,22,21,21,21,21)	(1,2,3,6,11,18,22,20,11,1)	95	22	8	78	20941	0%
(30,30,29,25,23,21,20,20,20,20)	(1,2,3,6,11,18,23,20,11,1)	96	23	7	77	15626	0%
(30,29,28,27,26,22,22,22,22,22)	(1,2,4,8,12,19,19,19,12,1)	97	19	11	81	11302	0%
(30,30,26,23,22,21,20,20,20,20)	(1,2,3,6,11,18,24,21,11,1)	98	24	6	76	7863	0%
(30,29,28,27,26,22,20,20,20,20)	(1,2,4,8,12,19,20,21,11,1)	99	21	9	79	5654	0%
(30,29,28,27,25,21,21,21,21,19)	(1,2,4,8,13,21,20,20,10,1)	100	21	9	79	3686	0%

1	2	3	4	5	6	7	8
(30,29,28,27,25,21,21,21,20,20)	(1,2,4,8,13,21,20,20,11,1)	101	21	9	79	2287	0%
(30,29,28,27,25,21,19,19,19,19)	(1,2,4,8,13,21,22,20,10,1)	102	22	8	78	1404	0%
(30,29,28,27,25,21,21,21,21,21)	(1,2,4,8,13,21,21,21,11,1)	103	21	9	79	802	0%
(30,29,28,27,24,20,20,20,20,19)	(1,2,4,8,14,23,22,20,10,0)	104	23	7	77	436	0%
(30,29,28,27,24,20,20,20,20,18)	(1,2,4,8,14,23,22,20,10,1)	105	23	7	77	195	0%
(30,29,28,27,24,20,19,19,19,19)	(1,2,4,8,14,23,23,20,10,1)	106	23	7	77	102	0%
(30,29,28,26,23,20,19,19,19,19)	(1,2,4,8,15,23,23,20,10,1)	107	23	7	77	54	0%
(30,29,28,27,24,20,20,20,20,20)	(1,2,4,8,14,23,23,21,11,1)	108	23	7	77	18	0%
(30,29,28,26,23,20,20,20,20,20)	(1,2,4,8,15,23,23,21,11,1)	109	23	7	77	4	0%
(30,29,28,24,21,18,18,18,18,18)	(1,2,4,8,15,25,25,19,10,1)	110	25	5	75	2	0%

2 Graphical algorithm for the Knapsack problem

2.1 Dynamic programming algorithm for the problem

This algorithm is based on Belman's principe of optimality.

Algorithm decides instances only for $a_i \in Z^+,\ i = 1, \dots, n,\ A \in Z^+$.

On each step $i = 1, \dots, n$ we calculate function

$$g_i(t) = max_{x_i \in \{0,1\}}(c_i x_i + g_{i-1}(t - a_i x_i)),$$

where $t \geq a_i x_i$ for each point $0 \leq t \leq A$. For each t we fix $x_i(t) = \arg\max g_i(t)$.

On the step $i = n$ in point $t = A$ we have optimal solution.

For example:

$$f(x) = 5x_1 + 7x_2 + 6x_3 + 3x_4 \longrightarrow \max$$

$$2x_1 + 3x_2 + 5x_3 + 7x_4 \leq 9$$

$$x_i \in \{0, 1\},\ i = 1, \dots, 4.$$

The computational results are presented in the following table:

t	$g_1(t)$	$x(t)$	$g_2(t)$	$x(t)$	$g_3(t)$	$x(t)$	$g_4(t)$	$x(t)$
0	0	(0,,,)	0	(0,0,,)	0	(0,0,0,)	0	(0,0,0,0)
1	0	(0,,,)	0	(0,0,,)	0	(0,0,0,)	0	(0,0,0,0)
2	5	(1,,,)	5	(1,0,,)	5	(1,0,0,)	5	(1,0,0,0)
3	5	(1,,,)	7	(0,1,,)	7	(0,1,0,)	7	(0,1,0,0)
4	5	(1,,,)	7	(0,1,,)	7	(0,1,0,)	7	(0,1,0,0)
5	5	(1,,,)	12	(1,1,,)	12	(1,1,0,)	12	(1,1,0,0)
6	5	(1,,,)	12	(1,1,,)	12	(1,1,0,)	12	(1,1,0,0)
7	5	(1,,,)	12	(1,1,,)	12	(1,1,0,)	12	(1,1,0,0)
8	5	(1,,,)	12	(1,1,,)	13	(0,1,1,)	13	(0,1,1,0)
9	5	(1,,,)	12	(1,1,,)	13	(0,1,1,)	13	(0,1,1,0)

The optimal solution is $(0, 1, 1, 0)$ and $f_{opt} = g_4(9) = 13$.
The run time is $O(nA)$.

2.2 Graphic algorithm

We describe a function $g_i(t)$ in the next table:

t	t_0	t_1	$\dots$	t_{m_i}
g	f_0	f_1	$\dots$	f_{m_i}

For $t \in [t_j, t_{j+1})$ we have $g_i(t) = f_j$.
Function $g_{i+1}(t)$ are constructed from $g_i(t)$:

$$g^1(t) = g_i(t),\ x_{i+1}(t) = 0,$$

$$g^2(t) = c_{i+1} + g_i(t - a_{i+1})\ x_{i+1}(t) = 1,\ a_{i+1} \le t,$$

$$g^2(t) = g^1(t),\ x_{i+1}(t) = 0,\ a_{i+1} > t,$$

$$g_{i+1}(t) = \max\{g^1(t), g^2(t)\}$$

.

i.e. $g^2(t)$ are constructed from $g_i(t)$ with dislocation upward on c_{i+1} and to the right on a_{i+1}. So we can describe $g^2(t)$:

t	$t_0 + a_{i+1}$	$t_1 + a_{i+1}$	$\dots$	$t_{m_i} + a_{i+1}$
g	$f_0 + c_{i+1}$	$f_1 + c_{i+1}$	$\dots$	$f_{m_i} + c_{i+1}$

Graphics for $g^1(t)$ and $g_i(t)$ are identical.

So to construct $g_{i+1}(t) = \max\{g^1(t), g^2(t)\}$ we must look less than $2m_i$ points $\{t_0, t_1, \dots, t_{m_i}, t_0 + a_{i+1}, t_1 + a_{i+1}, \dots, t_{m_i} + a_{i+1}\}$, which in $[0, A]$.

For $a_i \in Z,\ i = 1, \dots, n$ the number of such points is less or equal A. That's why we have run time less than $O(nA)$ for integer instances.

Let's consider computational results of Graphical algorithm for the instance.

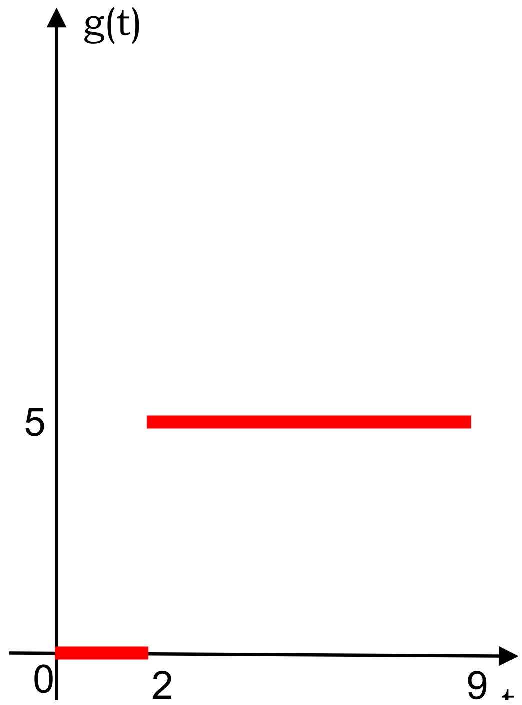

Рис. 5:

Step 1. Computational results and function $g_1(t)$ are presented on Fig.5. We save the following table:

t	0	2
g	0	5
$x(t)$	$(0,,,)$	$(1,,,)$

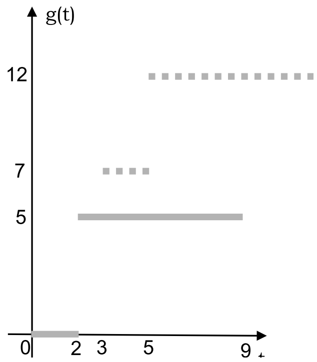

Рис. 6:

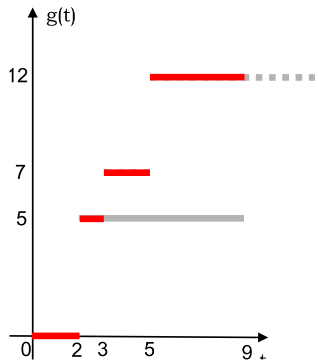

Рис. 7:

Step 2. On the Fig.6 function $g^1(t)$ and $g^2(t)$ are presented. To construct $g_2(t)$ we must look following points $0, 2, 0+3, 2+3$.

Computational results and function $g_2(t)$ are presented on Fig. 7.

We save the following table:

t	0	2	3	5
g	0	5	7	12
$x(t)$	$(0,0,,)$	$(1,0,,)$	$(0,1,,)$	$(1,1,,)$

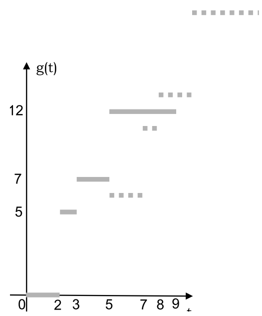

Рис. 8:

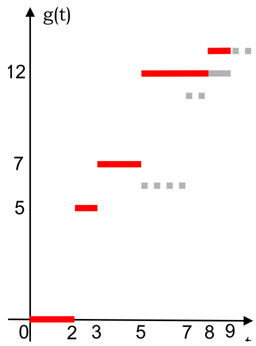

Рис. 9:

Step 3. On the Fig.8 function $g^1(t)$ and $g^2(t)$ are presented. To construct $g_3(t)$ we must look following points $0, 2, 3, 5, 0 + 5, 2 + 5, 3 + 5$. The point $5 + 5 > 9$ is not considered.

Заметно, что многие фрагменты $g^2(t)$ (обозначено пунктиром) "поглощаются" и не участвуют в $g_3(t)$. Computational results and function $g_3(t)$ are presented on Fig. 8.

We save the following table:

t	0	2	3	5	8
g	0	5	7	12	13
$x(t)$	$(0, 0, 0,)$	$(1, 0, 0,)$	$(0, 1, 0,)$	$(1, 1, 0,)$	(0,1,1,)

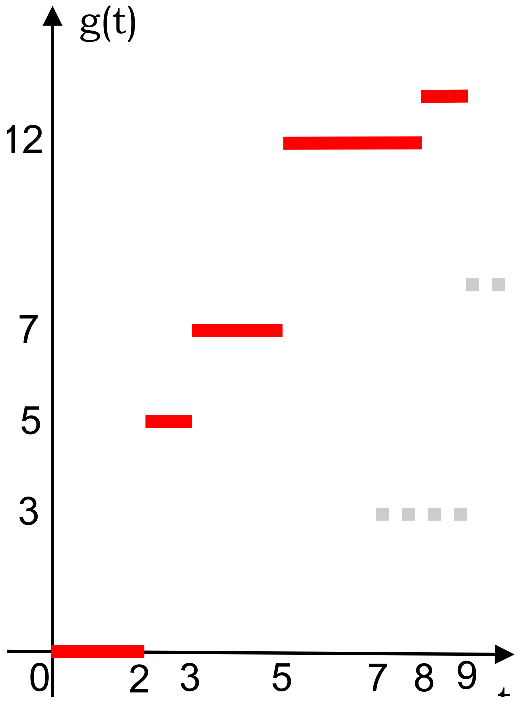

Рис. 10:

Step 4.

On the Fig.10 function $g^1(t)$ and $g^2(t)$ are presented. To construct $g_4(t)$ we must look following points $0, 2, 3, 5, 8, 0+7, 2+7$. Points $3+7, 5+7, 8+7$ are not considered.

Computational results and function $g_4(t)$ are presented on Fig. 10.

So we've looked only 6 points.

We save the following table:

t	0	2	3	5	8
g	0	5	7	12	13
$x(t)$	$(0, 0, 0, 0)$	$(1, 0, 0, 0)$	$(0, 1, 0, 0)$	$(1, 1, 0, 0)$	(0,1,1,0)

Not-integer instance

Let's consider the instance:

$$f(x) = 5x_1 + 7x_2 + 6x_3 + 3x_4 \longrightarrow \max$$

$$2.5x_1 + 3.001x_2 + 5.17x_3 + 7x_4 \leq 8.9$$

$$x_i \in \{0,1\},\ i = 1, \ldots, 4.$$

Step 1. We save the following table:

t	0	2.5
g	0	5
$x(t)$	$(0, , ,)$	$(1, , ,)$

Step 2.To construct $g_2(t)$ we must look following points $0, 2.5, 0 + 3.001, 2.5 + 3.001$.

We save the following table:

t	0	2.5	3.001	5.501
g	0	5	7	12
$x(t)$	$(0, 0, ,)$	$(1, 0, ,)$	$(0, 1, ,)$	$(1, 1, ,)$

Step 3. To construct $g_3(t)$ we must look following points $0, 2.5, 3.001, 5.501, 0{+}5.17, 2.5{+}5.17, 3.001{+}5.17$. The point $5.501{+}5.17 > 8.9$ is not considered.

We save the following table:

t	0	2.5	3.001	5.501	8.171
g	0	5	7	12	13
$x(t)$	$(0, 0, 0,)$	$(1, 0, 0,)$	$(0, 1, 0,)$	$(1, 1, 0,)$	(0,1,1,)

Step 4. We have analogous table.

The optimal solution is $(0, 1, 1, 0)$. The optimal value of objective function is 13.

2.3 Run times

In the instance with dynamic programming algorithm we've looked $4 * 9 = 36$ points and with Graphical algorithm only $2 + 3 + 7 + 6 = 18$ points.

Graphical algorithm return the exact solution for instances where $a_i \notin Z$, $\forall i$, $A \notin Z$ too.

On steps 3 and 4 we have that the number of points don't grow exponential. So we assume that run time of Graphical algorithm is polynomial for great part of instances.

We offer to enumerate x_i according to rule: $\frac{c_1}{a_i} \geq \frac{c_2}{a_2} \geq \ldots \geq \frac{c_n}{a_n}$. In this case the number of considered points is smallest.

The algorithm indirectly takes into account features of the problem. In Graphic algorithm on step 4 the function $g^2(t)$ has not influenced on $g_4(t)$, i.e. some heuristic feature of the task is taken into account.

Список литературы

[1] Пападимитриу Х., Стайглиц К. *Комбинаторная оптимизация. Алгоритмы и сложность*, М.: Мир, 1985, – 512 с.

[2] Keller H., Pferschy U., Pisinger D. *Knapsack problems*, Springer, 2004, – 546 p.

[3] Беллман Р. *Динамическое программирование.* - М.: ИЛ, 1960.

[4] Сигал И.Х., Иванова А.П. *Введение в прикладное дискретное программирование.* - М.: ФИЗМАТЛИТ, 2002, – 237 с.