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# Static scheduling research to minimize weighted and unweighted tardiness: A state-of-the-art survey

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**Abstract**

This paper reviews research on the total tardiness (TT) and total weighted tardiness (TWT) problems. Heuristic methods and optimizing techniques are surveyed for both types of problems in the single-machine environment. Complexity theory related to these problems is also discussed. Some extensions of the TT and TWT problems are given for multi-machine environments.



*Keywords:* Total tardiness; Total weighted tardiness; Single-machine scheduling; Multi-machine scheduling

## 1. Introduction

In the area of scheduling, two of the most researched problems are the single-machine total (mean) tardiness (TT) and total (mean) weighted tardiness (TWT) problems. Although the first research on the TT problem was done as early as 1961, the original problem and its weighted version remain challenging topics of ongoing research. In 1990, an excellent survey of research on the weighted tardiness problem was provided by Abdul-Razaq et al. (1990). This review was good and extensive, but it dealt only with the weighted tardiness problem in the single-machine system. The most recent survey was done by Koulamas (1994); however, reviewed the research on the TT problem only.

Although the papers by Abdul-Razaq et al. (1990) and Koulamas (1994) were of high quality, neither explored the relationship between the TT and TWT problems. However, during the past three decades, these two problems have become intertwined, with research on one of the problems encouraging additional research on the other problem. In fact, a number of researchers have considered solutions to the two problems within the same paper. Therefore, we feel it is relevant to review the literature on both problems simultaneously, especially since no researcher has done so recently. In addition, we present a survey of extensions of these problems to multi-machine environments.

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## 2. The problems

This paper reviews research on just two problems, namely, the Total Tardiness problem and the Total Weighted Tardiness problem; we will call them the TT problem and the TWT problem, respectively. Since mean tardiness and TT are different from one another only by a multiplicative constant, the review of research on mean tardiness is also included here. To define these two problems, we will borrow the assumptions and terminology used by Conway et al. (1967). In the single-machine environment, we have $n$ jobs, $1, 2, \ldots, n$, all ready at time 0, to be processed on a single machine which is never idle. No pre-emption of jobs is allowed. A job $j$ is defined by its processing time $p_j$ (which includes the set-up time of the job $j$), and its due date $d_j$. The single-machine Total Tardiness problem requires minimizing the total tardiness $T$ such that

$$T = \sum_{j=1}^{n} T_j,$$

where, for the processing sequence $(1, 2, \ldots, n)$, $T_j$, the tardiness of job $j$, is defined as

$$T_j = \max\left(0, \sum_{i=1}^{j} p_i - d_j\right).$$

Again, if $w_j$ represents the weight of job $j$, then the single-machine TWT problem minimizes $T$, where

$$T = \sum_{j=1}^{n} w_j T_j$$

and where $T_j$ is the same as above. The mathematical expressions for the TT and the TWT problems are more complex in multi-machine systems.

The material reviewed in this paper is organized in the following way. First, optimizing techniques for the TT problem are reviewed, followed by a survey of optimizing methods for the TWT problems that are presented in the literature. This is followed by the presentation of some extensions of the both TT and TWT problems in multi-machine environments. Next, heuristic methods available for the TT and TWT problems are discussed. We then discuss the complexity theory related to the combinatorial optimization problems and also mention how complex the TT and TWT problems are in terms of this theory. Finally, we conclude with some suggestions for future research.

## 3. Optimizing techniques: Single-machine TT problems

Most of the optimizing techniques developed to solve the TT problem used branch-and-bound or dynamic programming as the basic computational vehicle. Also, a few researchers used a combination of these two implicit enumeration techniques, whereas other researchers have proposed methods which do not use any specific controlled enumeration methodology.

Two of the earliest optimizing techniques for the TT problem employed the branch and bound method developed by Schild and Fredman (1961) and the dynamic programming method developed by Held and Karp (1962). Although these two controlled enumeration approaches represent a significant improvement over exhaustive search, they are still extremely laborious and are applicable only to relatively small problems.

Lawler (1964) presented a dynamic programming formulation for the TT problem. Although more efficient $O(n2^n)$ than explicit enumeration, the method was ‘‘computationally infeasible for even modest sized problems’’ in the early 1960s (p. 302). In the same paper, for the linear penalty function (TWT), Lawler presented an LP formulation requiring $n + 2T$ constraints where

$$T = \sum_{j=1}^{n} p_j.$$

It was shown that it is possible to arrive at a much smaller mixed integer linear programming formulation requiring only $2^n$ constraints, $3n$ continuous variables and $n(n-1)/2$ $0-1$ variables.

Elmaghraby (1968) presented a network model which was somewhat equivalent to the earlier dynamic programming formulations. The cost minimization objective function was changed to a shortest route problem. Elmaghraby (1968) also showed how a branch and bound method can be applied to substantially reduce the required computational time compared to the dynamic programming method. Elmaghraby (1968) did not report any computational experience but Shwimer (1972) later noted that the six step method of Elmaghraby (1968) would be difficult to program efficiently on a computer.

A good theoretical development for the TT problem was done by Emmons (1969) who developed some relationships among the job variables $p_j$ and $d_j$. By exploiting these relationships, the size of most problems can be reduced considerably. Emmons (1969) developed three basic theorems and a good number of corollaries relating job variables and used them to locate as many jobs as possible at the beginning and at the end of an optimal sequence. At the end of the paper, Emmons (1969) extended his results to the general criterion of the sum of identical, convex, non-decreasing functions of job tardiness (TWT) and then proposed an efficient algorithm for the same problem.

Emmons (1969) did not computerize his algorithm nor did he compare his work with any other procedure. Although his branching technique can be improved by some implicit enumeration method, Emmons (1969) was the first to explore the relationships among job variables to reduce the size of the problem. This approach was later adopted by many researchers to construct partial optimal sequences using the three theorems and then applying an implicit enumeration scheme for the remaining unsequenced jobs. After Emmons, some extensions to his theoretical results were made by Fisher (1976) and Rinnooy et al. (1975); however, these extensions were minor compared to the monumental work of Emmons.

The next optimizing technique was devised by Srinivasan (1971) who called his method a "three phase hybrid algorithm". The first two phases of this algorithm were used, as in Emmons (1969), to locate as many jobs as possible in the beginning and the end of the optimal sequence. Phase III then arranged the remaining jobs optimally using a modified dynamic programming algorithm.

Another branch-and-bound algorithm was developed by Shwimer (1972) to minimize total penalty cost due to job tardiness. Computational results were reported in the case of linear penalty functions and the modifications which permit treatment of non-linear functions were also described. Shwimer (1972) suggested the use of heuristics to reduce branching at the early stages of the search and emphasized the need for better initial bounds.

One branch-and-bound approach was proposed by Rinnooy Kan et al. (1975) who made some minor extensions to Emmons' (1969) results. Their algorithm was employed initially to solve the TWT problem and then to solve the TT problem as a special case thereof. They observed that the lack of structure which is typical of most combinatorial optimization problems is more pronounced for the TWT problem than for the TT problem. They suggested that looking into the effects of interchanging more than two jobs at a time may be worthwhile to determine precedence relationships among some jobs. One of the important observations they had was as follows: very sharp lower bounds are needed in the earlier stages of the search tree, whereas, simpler lower bounds could be more effective at latter stages.

Another branch-and-bound algorithm for the TT problem was presented by Fisher (1976) whose algorithm uses a "dual problem to generate a good feasible solution to start with, and an extremely sharp lower bound to the optimal solution" (p. 229). The author generated a Lagrangian problem which was subsequently used to generate the lower bound. Computational results for 50 problems were reported. The author developed some theoretical results which were used to reduce the number of enumerations at the branching stage. It was claimed that these theoretical findings contain the results of Emmons (1969) on the TT problem with some minor extensions.

The next branch-and-bound algorithm was devised by Picard and Queyranne (1978), who developed an algorithm for the well known traveling salesman problem (TSP), of which two special cases are the TT and TWT problems. The computational results for both the TT and TWT problems are reported, but no comparisons were made with existing methods.

Another solution for the TT problem, called the "chain algorithm", developed by Baker and Schrage (1978), used a dynamic programming scheme along with Emmons' (1969) results. Their technique was compared to the method developed by Rinnooy Kan et al. (1975) and was found to be much superior in terms of CPU time.

One of the best methods for the TT problem was devised by Schrage and Baker (1978) who made skillful use of the dynamic programming technique. The algorithm was compared computationally with those of Kan et al. (1975), Fisher (1976), Baker and Schrage (1978), Picard and Queyranne (1978). It was found that the algorithm of Schrage and Baker (1978) was by far the most efficient in terms of CPU time. In a study by Kao and Queyranne (1982), it was observed that a more recent dynamic programming algorithm of Lawler (1979) was superior to the DP algorithm of Schrage and Baker in terms of CPU time.

Two more dynamic programs for the TT problem are due to Potts and Van Wassenhove (1982, 1985), who utilized Lawler's (1977) decomposition algorithm (discussed in the next section) in combination with the Schrage–Baker algorithm to produce an algorithm capable of solving problems with as many as 100 jobs. Therefore, the authors legitimately concluded that, in practical terms, the TT problem had been solved.

Another algorithm for the TT problem, called MINIT, was devised by Sen et al. (1983) and was based on existing relationships among the job variables. The authors compared their algorithm with the DP algorithm of Schrage and Baker (1978) and observed that if storage availability is unlimited, DP must be preferred to MINIT; otherwise, MINIT may be considered as a viable alternative.

Another optimizing technique for the TT problem is due to Sen and Borah (1991), who developed a set of relationships among the job variables and a branching scheme based on these relationships. The computerized technique was compared with the DP algorithm of Schrage and Baker (1978). Although for most problems DP outperforms the Sen–Borah method, the latter method generates a smaller set of candidate sequences (at the branching stage) for each of the 120 randomly generated test problems.

More recently, several excellent papers on optimizing methods for the TT problem have appeared in the literature. One of these was by Chang et al. (1995). After noting that Potts and Van Wassenhove (1982) had presented some conditions on decomposition positions to improve the efficiency of Lawler's (1977) decomposition algorithm, Chang et al. (1995) obtained some new conditions on the left-most decomposition positions. Another excellent paper was due to Yu (1996), who showed that if a procedure of proper augmentation beginning from "null" were used, augmentation of a consistent partial order would produce a partial order which is itself consistent. Given his findings, Yu (1996) claimed to reduce the gap between Emmons' dominance theorems and the normal procedure of augmentation of partial orders. Szwarc and Mukhopadhyay (1996) obtained a new decomposition rule for the TT problem and used a pure decomposition approach to devise a fast branch-and-bound algorithm.

An altogether different approach to the single-machine TT problem was taken by Tansel and Sabuncuoglu (1997). The authors examine Emmons' (1969) theorems about precedence relationships between job pairs from a geometric perspective. They generated fresh insight into dominance properties among jobs so that certain classes of easy and hard instances could be more readily distinguished from each other.

In 1993, Koulamas reviewed a small subset of the single-machine TT problems, which are polynomially solvable. He also suggested extensions to some of these problems.

In the past few years, a good number of papers on dual criteria of which one is the single-machine TT problem have appeared in the literature. These include Davis and Kanet (1993), Ibaraki and Nakamura (1994), Vairaktarakis and Lee (1995), Ben-Daya et al. (1996), and Federgruen and Mosheiov (1997). The second criterion they considered include number of tardy jobs and earliness penalty.

All in all, the TT problem has been optimally solved by several researchers in an efficient manner, but there is still a possibility that more relationships among the job variables can be found and hence more efficient solution techniques for the TT problem may be developed by some future researchers.

Table 1 provides a brief summary of the important advancements in research towards solving the TT problem to optimality.

## 4. Optimizing techniques: Single-machine TWT problems

For the TWT problem, the objective function to be minimized is of the form

$$\sum_{j=1}^{n} w_j T_j.$$

Most of the research that has been done thus far for the TT problem has been extended to the TWT problem. Noteworthy papers include Elmaghraby (1968), Emmons (1969), Picard and Queyranne (1978), and Schrage and Baker (1978). For a special case of the problem, when the weighting of jobs is agreeable, i.e., when $p_i < p_j$ implies $w_i > w_j$, Lawler (1977) developed a "pseudo-polynomial" time dynamic programming algorithm for the TWT problem. This approach also solves the TT problem, which is a special case of the TWT problem with $w_i = 1$. Lawler's algorithm for the TWT problem has the worst case running time of $O(n^4 P)$ where $P = \sum p_j$. Lawler (1982) later developed another algorithm for the TT problem and called it a "fully polynomial approximation scheme" which is a modification on the "pseudo-polynomial" time algorithm mentioned above. The worst case running time for this method is $O(n^7/\varepsilon)$ where $\varepsilon > 0$ is a constant.

One paper that primarily dealt with the TWT problem is by Potts and Van Wassenhove (1991) who presented a branch-and-bound algorithm for

Table 1
Optimizing techniques for TT problems

| Year | Method | Complexity | TWT | Reference |
|---|---|---|---|---|
| 1961 | BB[a] | | | Schild and Fredman (1961) |
| 1962 | DP[b] | | | Held and Karp (1962) |
| 1964 | DP | $O(n2^n)$ | | Lawler (1964) |
| 1968 | Network model | | Yes | Elmaghraby (1968) |
| 1969 | Dominance rules | | Yes | Emmons (1969) |
| 1971 | DP-based hybrid | | | Srinivasan (1971) |
| 1972 | BB | 30 Jobs, $\leqslant 4.13$ s | | Shwimer (1972) |
| 1975 | BB | 20 jobs, $> 300$ s | Yes | Rinnooy Kan et al. (1975) |
| 1976 | BB/Lagrangian relaxation | | | Fisher (1976) |
| 1977 | Decomposition/DP | $O(n^4 \sum p_i)$ | Yes | Lawler (1977) |
| 1978 | Decomposition/DP | 50 jobs | | Baker and Schrage (1978) |
| | BB | 20 jobs, $\leqslant 12.8$ s | Yes | Picard and Queyranne (1978) |
| | DP | 50 jobs, 0.844 s | | Schrage and Baker (1978) |
| 1979 | | | | |
| 1982 | Decomposition/DP | $O(n^7/\varepsilon)$ | | Lawler (1982) |
| | Lawler decomposition extension | | | Potts and Van Wassenhove (1982) |
| 1983 | MINIT | | | Sen et al. (1983) |
| 1987 | DP | $\leqslant 100$ jobs | | Potts and Van Wassenhove (1987) |
| 1991 | DP | | | Sen and Borah (1991) |
| 1995 | Decomposition extension | | | Chang et al. (1995) |
| 1996 | Extension of Emmons' dominance | | | Yu (1996) |
| | Decomposition/BB | 100–150 jobs | | Szwarc and Mukhopadhyay (1996) |
| 1997 | Decomposition | | | Tansel and Sabuncuoglu (1997) |

[a] BB = Branch-and-bound.
[b] DP = Dynamic programming.

the problem. This method used a Lagrangian relaxation approach to generate sharp lower bounds for the TWT problem and the associated sub-problem of minimizing total weighted completion time. Although the method was branch-and-bound, it used dynamic programming for checking dominance relationships among jobs in the search tree. Also, the method contradicted the popular idea that one should restrict the size of the search tree as much as possible by using the sharpest possible bounds. (Rinnooy Kan et al. (1975) had a similar observation more than a decade ago.) The so-called "curse of dimensionality" is somewhat avoided in this method since it does not need to store all the results generated in the process permanently.

Another paper that deals with the TWT problem is by Rachamadugu (1987) who identified a condition characterizing the location of two adjacent jobs in an optimal sequence of the TWT problem. No implicit enumeration scheme was presented nor was any computational experimentation conducted. The author proved that the condition mentioned above can be utilized in reducing the branching effort for a branch-and-bound technique. The author claimed that his rule has the potential to identify the situations where the first job in an optimal sequence of the TWT problem is known even before the problem is fully solved.

Szwarc and Liu (1993) considered Arkin and Roundy's (1991) (discussed in the next section) weighted TT problem where tardiness penalties are proportional to the processing times. They presented a decomposition mechanism, which either solves the problem completely or decomposes the problem into smaller sub-problems.

Recently, a new lower bounding scheme for the single-machine TWT problem was developed by Akturk and Yildirim (1998). They introduced a dominance rule, which guarantees a sufficient condition for a local optimum for the TWT problem. They claimed that if the proposed rule is violated by any sequence, then exchanging the violating jobs will either reduce TWT or leave it unaffected. Thus, the proposed rule can be used in decreasing the number of alternatives to be evaluated to find the optimal sequence. A brief summary of the research with TWT problems is given in Table 2.

## 5. Extensions to multi-machine systems

We have already reviewed the research in the TT and the TWT problems in the single-machine environment fairly extensively. Now, let us briefly look at a few papers on the TT and TWT problems for multi-machine environments. Both optimal and heuristic approaches are considered here. One technique was devised by Baker and Kanet (1983) who utilized a different definition of due date, called the modified due date (MOD), which is used in a dispatching rule for the TT problem in a multi-machine job shop system. The authors claimed that the MOD rule compares very favorably with other well-known dispatching rules that are used to solve the mean tardiness problem. Raman and Talbot (1993) presented an integer programming formulation of the TT problem in the job shop environment. The authors used the concept of MOD introduced in Baker and Kanet (1983) to develop a heuristic solution for the job shop TT problem.

Another paper considering minimizing TT in the two-machine flowshop was due to Sen et al. (1989). They developed a branch-and-bound solution procedure to reach the optimal solution.

Table 2
TWT problem

| Year | Method | Reference |
|---|---|---|
| 1968 | Network model/BB[a] | Elmaghraby (1968) |
| 1969 | Dominance rules | Emmons (1969) |
| 1975 | BB | Rinnooy Kan et al. (1975) |
| 1977 | DP[b] | Lawler (1977) |
| 1978 | Branch-and-bound | Picard and Queyranne (1978) |
| 1985 | BB/Lagrangian relaxation | Potts and Van Wassenhove (1985) |
| 1987 | Dominance rules | Rachamadugu (1987) |
| 1993 | Decomposition | Szwarc and Liu (1993) |
| 1998 | Dominance rules | Akturk and Yildirim (1998) |

[a] BB = Branch-and-bound.
[b] DP = Dynamic programming.

Three well-known heuristic methods, viz., the SPT rule, the EDD rule and the minimum slack time rule (MST) were tested and it was found that the SPT rule, with a little modification, performed the best among the heuristics. An optimizing algorithm due to branch-and-bound and the modified SPT heuristic were compared on a set of randomly generated problems.

Gelders and Sambandam (1978) presented four heuristics to minimize the sum of weighted tardiness and flowtime which is actually a bicriterion problem in the flowshop environment. Ow (1985) introduced a scheduling technique for the TWT problem for the proportionate flowshop system. In another paper Kim (1993) presented several heuristics for the flowshop scheduling problem with the objective of minimizing mean tardiness.

Koulamas (1996) examined a special case of the TT problem for the two-machine flowshop where he assumed that the first machine handles pre-processing while the second machine is dedicated to the main operation. It was also assumed that pre-processing time exceeds time spent on the main operation for each job. Some dominance conditions on jobs were presented and a polynomial-time heuristic for the problems was developed.

Arkin and Roundy (1991) developed a pseudo-polynomial-time algorithm similar to Lawler (1977) to solve the weighted tardiness problem in the parallel machine environment under the assumption that the weight of the specific job is proportional to its processing time. Recently, a polynomial decomposition heuristic was obtained by Koulamas (1997a, b) for the parallel machine TT problem. The proposed approach represented an extension of the decomposition principle embedded in the single-machine problem. The subproblems that were produced by the decomposition method were solved by a heuristic. A hybrid simulated annealing heuristic was also discussed.

Azizoglu and Kirca (1998) proposed a branch-and-bound algorithm for the NP-hard problem minimizing TT for identical parallel machines. Computational experimentation indicated that optimal solutions were found within a reasonable length of time for problems up to 15 jobs. The authors extended properties characteristic of the optimal schedule's structure to the uniform parallel machine environment.

Alidaee and Rosa (1997) presented a modified due date algorithm for scheduling a set of $n$ jobs on $m$ parallel machines to minimize total weighted and unweighted tardiness. Li and Cheng (1994) developed a method to determine an optimal job schedule to minimize maximum weighted absolute tardiness for parallel machine settings. The authors proved that this problem is NP-complete for the single-machine case and strongly NP-complete for the general case. They also discussed a polynomial-time heuristic for the problem.

Bernardo and Lin (1994) proposed an interactive procedure for a bicriterion problem on non-identical parallel machines of which one criterion was TT (and the other was set-up costs). Since it did not utilize a weighted objective function, the procedure allowed the evaluation of job assignments in view of the current situation—without pre-emption due to previously assigned criterion weights.

A more recent paper is due to Barman (1998) who investigated the impact of a few combinations of some priority rules on lateness and tardiness for the three-stage flowshop. The performance criteria included: mean tardiness, maximum tardiness, mean lateness and percent of tardy jobs. Of the priority rules considered in this paper, the most effective were the Modified Shortest Processing Time, Shortest Processing Time and Earliest Due Date rules.

Another recent paper is due to Singer and Pinedo (1998) who studied a number of branch-and-bound algorithms to minimize TWT in job shops. An analysis of precedence constraints provided the basis for the bounding schemes. The authors obtained optimal solutions for all the cases where ten jobs and ten machines are considered.

## 6. Heuristics

Although the initial attempts at solving the TT and TWT problems used optimizing methods based on implicit enumeration techniques, a variety of heuristics have also been reported in the literature. One of these is the shortest processing time (SPT) rule which sequences the jobs in non-decreasing order of $p_j$, with ties broken in the non-decreasing order of $d_j$. This rule provides an optimal solution, if it is impossible for any job to be on time in any sequence (1974). Another heuristic is the well known earliest due date (EDD) rule which schedules all the jobs in the non-decreasing order of $d_j$ (with ties broken in the SPT order). This rule is also optimal, if the EDD sequence produces no more than one tardy job (Baker, 1974).

The algorithm that was developed by Wilkerson and Irwin (1971) for the TT problem, was the first heuristic that tried to utilize relationships between the job variables $p_j$ and $d_j$. Baker and Martin (1994) later compared the performance of several algorithms for the TT problem and discussed conditions under which the Wilkerson–Irwin (W–I) heuristic is able to provide optimal solutions.

Another efficient heuristic method was developed by Fry et al. (1989). This method utilized the adjacent pairwise interchange (API) method to minimize mean tardiness. Fry et al. (1989) considered 192 randomly generated problems and compared their API method with the W–I method. Their results indicated that the performance of the API method was much superior to the W–I method in terms of solution quality.

Potts and Van Wassenhove (1991) noted that the optimizing algorithms based on implicit enumeration require considerable computer resources in terms of both CPU time and storage requirements. In some situations, these huge computer resources may be neither available nor warranted. With this in mind, they tested a number of quick and effective heuristics (like SPT, EDD and some variations and combinations of these) against each other and against the optimal solutions for the problems tested. They concluded that the W–I heuristic (1971) outperformed the rest. Although the heuristic of Fry et al. (1989) was published earlier, it was not tested by Potts and Van Wassenhove against W–I or other heuristics.

A more recent heuristic for the TT problem was developed by Holsenback and Russell (1992). It utilized the dominance criteria of Emmons (1969) along with a heuristic developed by them. This heuristic used net benefit of relocation (NBR) analysis to determine which job must be placed last in a sequence to reduce TT. They conducted computational experimentation which indicated that the NBR heuristic was superior to the API method of Fry et al. (1989) both in solution quality and in CPU time. They also compared their heuristic with the optimizing algorithm of Potts and Van Wassenhove (1987) and claimed that NBR performs well compared to the Potts–Van Wassenhove algorithm, especially for large problems.

Another recent heuristic to minimize mean tardiness in the single-machine system was developed by Panwalkar et al. (1993). Their algorithm (PSK) was compared with the W–I (Wilkerson and Irwin, 1971), API (Fry et al., 1989), and the HR (Holsenback and Russell, 1992) heuristics for a wide range of problems and was found to be superior to each of them.

More recently, Fadlalla et al. (1994) developed a simple heuristic in which the most promising job to sequence in the last position is found recursively. This heuristic was tested on classes of problems known to be difficult to solve. Fadlalla et al. (1994) claimed that their heuristic outperformed the W–I heuristic. They also demonstrated that their heuristic's performance improved as problem size increased.

Ben-Daya and Al-Fawzan (1996) developed a simulated annealing approach to solve the single-machine mean tardiness problem. They claimed that their heuristic produced much better solutions than the heuristics of Fry et al. (1989) and Holsenback and Russell (1992).

The modified due date rule (MDD) is known to be an efficient heuristic that can minimize TT in the single-machine environment. Alidaee and Gopalan (1997) showed that the PSK rule of Panwalkar et al. (1993) is nothing but an implementation of the MDD rule. They also offered some new insight on the relationship between the MDD rule and the heuristic of Wilkerson and Irwin (1971).

In an earlier paper, Panwalker et al. disputed the experimental results of Holsenback and Russell (1992) who claimed that their NBR heuristic for the single-machine TT problem had better solution quality than the API routine of Fry et al. (1989). Panwalkar et al. (1993) claimed that their PSK heuristic outperformed the HR heuristic. Russell and Holsenback (1997a, b) contradicted that finding and presented experimental results which showed that, in general, the PSK heuristic was inferior to the NBR heuristic. Also, in another paper, Russell and Holsenback (1997a, b) improved on their earlier NBR heuristic with a modified version which they called the M-NBR heuristic. In comparison with the NBR heuristic, the M-NBR heuristic provided better solution quality without any increase in computational effort.

Rachamadugu (1987) supplemented the dominance tests developed by Rinnooy Kan et al. (1975) by obtaining a local precedence relationship among adjacent jobs in an optimal sequence for the TWT problem. The author also described a sufficiency condition for identifying the initial job in an optimal sequence without solving the entire problem.

More recent heuristics for the TWT problem were obtained by Crauwels et al. (1998) who compared such local search techniques as tabu search, simulated annealing, descent, threshold search and genetic algorithms by utilizing both permutation and binary representations of actual sequences for a large set of test problems. It was found that permutation methods were more likely to produce an optimal solution than binary-based methods, but binary-based methods consistently generated good quality solutions whereas permutation-based methods did not. Of the search methods that were tested, tabu search was found to be superior to the others.

Recently, a good number of heuristics have been presented in the literature which dealt with dual criteria of which one was the total (or mean) tardiness problem. Some of these papers include Fadlalla (1994), Federgruen and Mosheiov (1994), Sabuncuoglu and Gurgun (1996), Sridharan and Zhou (1996), Li (1997), and James and Buchanan (1998).

## 7. The complexity theory

It is well known that scheduling problems may be classified into two different groups according to the theory of computational complexity of the combinatorial optimization problems (Lenstra and Rinnooy Kan, 1980). Complexity theory identifies a class of problems called NP-complete problems with two important properties:

(i) No NP-complete problem can be solved by an algorithm whose running time is a polynomial function of problem size.
(ii) If two problems are categorized as NP-complete, and if for one of them, it is found that the running time is a polynomially bounded function of the problem size, it will be found for the second problem also.

In addition, the combinatorial optimization problems that are found to be at least as difficult as a known NP-complete problem are called NP-hard. If a problem is not NP, it is a polynomially bounded or P problem. It is well known that the single-machine TWT problem is NP hard (Lenstra and Rinnooy Kan, 1980). For the TT problem, the question of computational complexity was open for a fairly long time (more than two decades), and only a few years ago, Du and Leung (1990) showed that the TT problem in the single-machine environment is also NP-hard. Consequently, large problems ($n > 100$) cannot be solved readily to arrive at the optimal solution due to the exponential or factorial growth in CPU time and storage requirements with increases in problem size.

## 8. Conclusion

It has been almost five decades since research on machine scheduling first appeared in the literature. The initial emphasis was on minimizing total flowtime (Smith, 1956), tardiness penalties (Elmaghraby, 1968; Emmons, 1969; Fisher, 1976) or variance of completion times (Eilon and Chowdhury, 1977). A good discussion of different aspects of production scheduling is available in Eilon (1979). It has been almost as long since the first papers on the TT problem (minimizing total tardiness on a single machine) were published. While the earliest research described optimizing techniques for solving the TT problem, heuristic methods, which afforded quick, efficient and near optimal solutions, eventually became a topic of intense investigation by late 1960s.

From complexity theory, it is known that for either the TT or TWT problem, solution time will increase exponentially—and, in most cases, factorially—as the problem size increases. Research done by Lenstra and Rinnooy Kan (1980) and Du and Leung (1990) proved that optimizing techniques for the TT and TWT problems are NP complete. It is therefore not surprising that recent research on these problems has placed greater emphasis on heuristics than on optimizing techniques. Future research should continue this trend; if good heuristics can be used to generate efficient initial solutions, then enormous improvement on the total solution time for the TT or TWT problem can be achieved.

Extension to multiple machines have been considered in jobshop, flowshop, and parallel machine environments. Due to computational complexity, only heuristics are presented as practical methods for finding acceptable solutions for these problems.

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