Dichotomy for sheduling problem $P_m|prec; p_j = 1|C_{max}$

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1. Statement of the problem

Set V of unit length tasks, partial order $\lessdot$ on V, and a deadline $d \in N$ [1]. Can V be scheduled on m processors so as to satisfy the precedence constraints and meet the overall deadline D, i.e. is there a schedule $\sigma : V \to \{1, 2, \ldots, d\} = D$ that
1) if $i \lessdot j$ then $\sigma(i) < \sigma(j)$;
2) for all $k \in D$ exist at most m jobs $i \in V$ that $\sigma(i) = k$.

Denote Σ_d the set of all schedules. The problem consist in minimization completion time , i.e.

$$\max_{i \in V} \sigma(i) \to \min_{\sigma \in \Sigma_d}. \qquad (1)$$

This problem can be interpreted as follows. Let G be an acyclic directed graph defining the precedence relation $\lessdot$ on V . The goal is to place vertices from G in horizontal rectangle of width m in such of way that the precedence between vertices ere satisfied and minimum number of columns is used.

2. Polyhedral relaxation

Let us assign a vector $x = (x_{ik}, i \in V, k \in D) \in R^{nd}$ to a schedule $\sigma : V \to D$ as follows:

$$x^{\sigma}_{ik} = \begin{cases} 1, & \sigma(i) = k, \\ 0, & \text{otherwise}. \end{cases} \qquad (2)$$

Let $M_d \in R^{nd}$ be the set of vectors satisfying system (3)-(7). On the basis of simple observations we can distinguish the following sets: let $V_k \in V$ be the set of vertices such that for all $i \in V \setminus V_k$ $x_{ik} = 0$ holds; let the set $D_i \in D$ for $i \in V$ be such that for all $k \in D \setminus D_i$ $x_{ik} = 0$.

$$\sum_{k \in D_i} x_{ik} = 1,\ i \in V; \qquad (3)$$

$$\sum_{i \in V_k} x_{ik} \le m,\ k \in D; \qquad (4)$$

$$x_{ik} \le \sum_{l \in D_j,\, l > k} x_{jl},\ (i, j) \in E(G),\ k \in D_i; \qquad (5)$$

$$0 \le x_{ik} \le 1,\ i \in V,\ k \in D_i; \qquad (6)$$

$$x_{ik} = 0,\ i \in V,\ k \in D \setminus D_i. \qquad (7)$$

Constraints (3) enforce that each job is started exactly once in time interval $[k; k + 1)$; inequalities (4) enforce that no more than m jobs can be processed in unit of time; inequalities (5) represent the precedence between jobs.

Proposition 1. *The integer vertices of polyhedron (3)-(7) and only they correspond to schedules in the sense of (2).*

Denote $M_{d,Z} = conv(M_d \setminus Z^{nd})$ the convex hull of all points corresponding to schedules. Consider the problem (1) as problem of integer linear programming on polyhedra M_d with objective function:

$$h(x) = \sum_{k=1}^{d} \lambda_k \sum_{i \in V_k} x_{ik} \to \min_{x \in M_d}, \qquad (8)$$

Here $\lambda_1 \ll \lambda_2 \ll \ldots \ll \lambda_d$.

Optimum values of problems (1) and (8) coincide if coefficients λ_k, $k = 1, \dots, d$ satisfy to conditions $m \sum_{\ell=1}^{k} \lambda_\ell \le \lambda_{k+1}$, $k = 1, 2, \dots, d-1$ [1].

3. Valid inequalities

The inequality $ax \le a_0$ is called valid for $M_{d,Z}$ if for all $x \in M_{d,Z}$ holds $ax \le a_0$. If there exists such $\bar{x} \in M_{d,Z}$ that $a\bar{x} = a_0$ then inequality $ax \le a_0$ is called a support inequality.

In [2] it has been shown that the inequality

$$\sum_{s=1}^{t} x_{i_s k} \le 1 \tag{9}$$

is a valid inequality for $M_{d,Z}$. Here $P \subset G$ be a path, $k \in D$.

Also in [2] it has been shown that the inequality

$$\sum_{l=1}^{k-1} x_{zl} + \sum_{j \in V(P)} x_{jk} + \sum_{l=k+1}^{d} x_{il} \le 1 \tag{10}$$

is a valid inequality for $M_{d,Z}$. Here $P \subset G$ be a path, $k \in D$, $i \in V(P)$ is the first vertex of P, $z \in V(P)$ is the last vertex of P.

Inequalities (9) and (10) have laid down in a basis of algorithm of cutting plane SHP.

4. Dichotomy for solving problem $P_m|prec; p_j = 1|C_{max}$

At the decision of a problem difficulty arises (8) methods of integer linear programming. It is connected with exponential growth of factors of criterion function. It can be avoided using a dichotomy. Application of a dichotomy for value search becomes possible thanks to inclusion $\Sigma_d \subset \Sigma_{d'}$ under condition of $d < d'$.

The initial interval of division gets out of inequalities $|P_{\max}| + 1 \le d_{\min} \le p_m$ proved in [1]. Here $|P_{\max}|$ – the length of maximal path in G, p_m – the density of G relatively m. Check not emptiness of set Σ_d is carried out by the decision of a problem of integer linear programming by algorithm SHP from any objective function.

Algorithm DMN serching d_{min}.
Input: $\mathbf{M}_d$ – polyhedral (3)-(7), $A = |P_{max}| + 1$, $B = p_m$.
Iteration k, $k = 1, 2, \dots$
Step 1. To find $d = \lceil \frac{A+B}{2} \rceil$.
Step 2. To solve task $\min\{h(x) | x \in \mathbf{M}_d \cap \mathbf{Z}^{nd}\}$ with algorithm SHP, where $h(x)$ any objective function.
If $\mathbf{M}_d \cap \mathbf{Z}^{nd} = \emptyset$ then $A = d$, $B = B$ and step 3.
If $\mathbf{M}_d \cap \mathbf{Z}^{nd} \ne \emptyset$ then $A = A$, $B = d$ and step 3.
Step 3. If $B - A = 1$ then quit: $d_{min} = B$. Otherwise iteration $(k+1)$.
End.

The given algorithm has been approved in numerical experiment. Exact decisions of the problems which dimension changes from 10 to 120 have been received.

References

[1] R. Yu. Simanchev and I. V. Urazova, *Integer model for scheduling problem of minimization of the general holding time of requirements with precedences by parallel machines* . Avtomatika and telemehanika, N 10, 2010.

[2] R. Yu. Simanchev and I. V. Urazova, *Polytope schedules of service of identical requirements parallel machines* . DAOR, N 1, v.18, 2011.