Structure and adjacency of vertices of b-factors polytope relaxation

R. Yu. Simanchev,

Omsk State University, osiman@rambler.ru

1. Introduction

In a number of algorithms of the decision of extreme combinatorial problems the information about polyhedral structure of convex hull of feasible solutions is widely used. For example, the full or partial linear description of a polytope of a problem allows to apply the device of linear and integer programming to its decision [1]; diameter of a polytope can serve as an estimation of number of iterations of the best simplex procedure [2]; classes of adjacent vertices are useful by working out of algorithms of local optimization, the decision of problems of identification of valid inequalities, revealing of conditions polynomial resolvability of problems [3].

In the present work the polytope of b-factors of any ordinary graph is considered. It is given the combinatorial description of nonintegral vertices and criterion of their adjacency concerning a relaxation of b-factors polytope.

Let G is an ordinary graph without loops and multiple edges with nodes set V and edges set E. We would will set a vector $b = (b_u, u \in V)$ with positive integer components. The subgraph $H \subseteq G$ is called as the b-factor, if $d_H(u) = b_u$ for all $u \in V$. Set of all b-factors of the graph G we will designate through $\mathcal{H}_b$.

To set of edges of the graph G it is comparable $|E|$-dimensional Euclidean space $\mathbb{R}^{\mathbb{E}}$ by means of one to one correspondence between elements of set and coordinate axes of space $\mathbb{R}^{\mathbb{E}}$. For any subgraph $F \subseteq G$ we will define its incidence vector $x^F \in \mathbb{R}^{\mathbb{E}}$ by a rule: $x^F_e = 1$ if $e \in EF$ and $x^F_e = 0$ otherwise. The b-factors polytope is the set

$$P_b = conv\{x^H \in \mathbb{R}^{\mathbb{E}} | H \in \mathcal{H}_b\}.$$

With the graph G we will connect its node-edge incidence matrix A. Assume

$$M_b = \{x^H \in \mathbb{R}^{\mathbb{E}} | Ax = b, 0 \leq x \leq 1\}.$$

The incidence vector of any b-factor satisfies to constraints of polyhedron M_b. Hence, the set M_b it is possible to consider as a polyhedral relaxation of a polytope P_b. In addition, as polyhedron M_b is a subset of an identity cube in space $\mathbb{R}^{\mathbb{E}}$, then $vertP_b \subseteq vertM_b$.

2. Graph structure of nonintegral vertices

In general, $P_b \neq M_b$ (in [4] the examples of nonintegral vertices of polyhedron M_b are redused). Thus it is important to notice that any integer vertex of polyhedron M_b is an incidence vector of the b-factor.

Let $x \in M_b$. With a point x we will connect following subgraphs of the graph G:

C_x - the fractional graph of a point x - is induced by set of edges $EC_x = \{e \in E | 0 < x_e < 1\}$;

T_x - the units graph of a point x - is induced by set of edges $ET_x = \{e \in E | x_e = 1\}$.

In terms of these graphs the structure of nonintegral vertices of polyhedron M_b can be described as follows.

Theorem 1. *The point $\bar{x} \in M_b$ is vertex, if and only if it is integer, or its fractional graph $C_{\bar{x}}$ and units graph $T_{\bar{x}}$ satisfies to following conditions:*

i) $C_{\bar{x}}$ is a union of even number of simple cycles without common nodes, and for any $e \in EC_{\bar{x}}$ takes place $\bar{x}_e = \frac{1}{2}$;

ii) $d_{T_{\bar{x}}}(u) = b_u - 1$ for all $u \in VC_{\bar{x}}$ and $d_{T_{\bar{x}}}(u) = b_u$ for all $u \in V \setminus VC_{\bar{x}}$.

3. Adjacency of vertices of polyhedron M_b

Let x^1,$x^2 \in M_b$ - pair of various points. Following designations will be necessary for us:

$$R(x^1, x^2) = \{e \in E | x_e^1, x_e^2 \in \{0, 1\}, x_e^1 + x_e^2 = 1\},$$

$$U(x^1, x^2) = \{u \in V \backslash V(C_{x^1} \cup C_{x^2}) | \delta_{T_{x^1}}(u) = \delta_{T_{x^1}}(u)\},$$

G_{x^1,x^2} - components of connectivity of the graph, that induced by set of edges $R(x^1, x^2)$, not containing nodes from $V(C_{x^1} \cup C_{x^2})$.

For $W \subseteq V$ we will put $\overline{W} = V \setminus W$.

Theorem 2. *Let x^1,$x^2 \in M_b$ - vertices, $R(x^1, x^2) = \emptyset$. Then x^1 and x^2 are adjacent in M_b, if and only if*

$$|V(C_{x^1} \cup C_{x^2})| = |E(C_{x^1} \cup C_{x^2})| - 1.$$

Theorem 3. *Let x^1,$x^2 \in M_b$ - vertices, $R(x^1, x^2) \neq \emptyset$. If x^1 and x^2 are adjacent in M_b, then*

$$|\overline{V(C_{x^1} \cup C_{x^2})}| \leq |R(x^1, x^2)| + |U(x^1, x^2)| \leq$$

$$\leq |\overline{V(C_{x^1} \cup C_{x^2})}| + 1$$

and $C_{x^1} \cup C_{x^2}$ is a set of simple cycles without common nodes.

Theorem 4. *Let x^1,$x^2 \in M_b$ - vertices, $R(x^1, x^2) \neq \emptyset$ and $C_{x^1} \cup C_{x^2}$ is a set of simple cycles without common nodes. Following statements are true:*
1) if $|R(x^1, x^2)| + |U(x^1, x^2)| = |\overline{V(C_{x^1} \cup C_{x^2})}| + 1$, then vertices x^1 and x^2 are adjacent in M_b if and only if the graph G_{x^1,x^2} either is empty, or is pair simple odd cycles with one common node;
2) if $|R(x^1, x^2)| + |U(x^1, x^2)| = |\overline{V(C_{x^1} \cup C_{x^2})}|$, then vertices x^1 and x^2 are adjacent in M_b, if and only if G_{x^1,x^2} is even cycle.

4. Conclusion

Direct consequence of the theorem 1 is the following: if the graph G doesn't contain even simple cycles, then polyhedron M_b is integer and, hence, coincides with P_b. From here follows that the condition of absence in G odd cycles is sufficient for polynomial resolvability of a weighed b-factor problem.

Now about an adjacency. If on points $x^1, x^2 \in M_b$ to impose a condition integrality we get to a situation $R(x^1, x^2) \neq \emptyset$. Besides, G_{x^1,x^2} will be in accuracy the graph induced by set of edges $R(x^1, x^2)$. From here we receive

Consequence. *Let $H_1, H_2 \in \mathcal{H}_b$. If $EH_1 \triangle EH_2$ induces in G either a simple even cycle, or pair of odd cycles with one common node, then in a polytope P_b vertices x^{H_1} and x^{H_1} are adjacent.*

It is obvious that this sufficient condition of an adjacency becomes also necessary, if G doesn't contain odd simple cycles.

References

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