Structural decomposition in projection methods for variational inequalities

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The variational inequality problem consists in finding a vector $x^* \in X$ such that

$$G(x^*)(x - x^*) \geq 0, \quad \forall x \in X, \qquad (1)$$

where $G : \Omega \to \mathbb{R}^n$ is a continuous mapping, $\Omega \subseteq \mathbb{R}^n$ is a open set, $X \subset \Omega$ is a nonempty convex closed set. Denote the problem (1) as $VI(G, X)$ and its solution set as X^* which we assume to be nonempty.

There are many applications of $VI(G, X)$ in economics, physics, operation research and this is one of main motivations to study it. Typically, these problems describe large-scale models with complex structure which are hard to solve by majority of existing methods (e.g.[1]). Computational problems are connected usually with the necessity to solve complicated auxiliary tasks on the each iteration. Now high performance multiprocessor computers can be used to solve such problems. Therefore the construction of iterative schemes that able to be parallelizable is very urgent today. The projection-type methods are very promising for development of parallel computation techniques.

It is known that a vector $x^* \in X^*$ if and only if x^* is a solution of projection equation $x^* = \pi_X(x^* - \lambda G(x^*))$ for any $\lambda > 0$, where $\pi_X(y) = \mathrm{argmin}\{\|y - x\| : x \in X\}$ is a projection of the vector y onto X. This property gave rise a lot of iterative schemes which include calculations of projection. Up-to-date survey about these methods is contained in [2].

In general an iterative scheme for solve $VI(G, X)$ can be wrote as

$$x^{k+1} = x^k + d(x^k, \lambda_k, \mu_k), \quad k = 0, 1, 2, \ldots, \qquad (2)$$

where $d(x^k, \lambda_k, \mu_k)$ is a descent direction for the function $\|x - x^*\|$ at x^k where $x^* \in X^*$ and λ_k, $\mu_k > 0$ are stepsizes. These methods differ from each other in the choice of a direction $d(x^k, \lambda_k, \mu_k)$ and in convergence properties of the process (2).

If $d(x^k, \lambda_k, \mu_k) = \pi_X(x^k - \frac{\lambda_k}{\mu_k} G(x^k)) - x^k$, $\mu_k = \|G(x^k)\|$, then the process (2) is a classical projection method for $VI(G, X)$. It's convergence is under following assumptions (e.g. [3]):

- acute angle condition: $G(x)(x - x^*) > 0$ for all $x \in X \backslash X^*$, $x^* \in X^*$;

- stepsize condition: $\sum_{k=0}^{\infty} \lambda_k = \infty$, $\sum_{k=0}^{\infty} \lambda_k^2 < \infty$.

If $d(x^k, \lambda_k, \mu_k) = \pi_X(x^k - \mu_k G(\bar{x}^k(\lambda_k))) - x^k$, $\bar{x}^k(\lambda_k) = \pi_X(x^k - \lambda_k G(x^k))$, then the process (2) is called extragradient method for $VI(G, X)$. There are many convergence theorem for this method. Common assumption can be formulated as following:

- G is pseudomonotone on X that is for any $x, y \in X$ $G(x)(y - x) \geq 0$ implies $G(y)(y - x) \geq 0$;

- stepsizes λ_k and μ_k are defined dynamically such the process (2) to converge to the set X^* (e.g. [2] and references there).

The simplicity of iterative scheme and possibility for modification make projection methods attractive for use. The most of studies are devoted to the choice of rules for stepsize parameters for acceleration convergence of the process (2). Such problem as how to calculate a projection on a set has poor attention. In general a quadratic optimization problem should be solved and this is main computational task. The situation is complicated for extragradient method because finding values of parameters λ_k and μ_k involves multiple projection on X. But there are simple sets (hyperplane, half-space, sphere, simplex and etc.) for which projection operations are easily performed. For such sets projection methods are popular.

One way to overcome the difficulties of practical implementation of the projection operation is to represent the feasible set of X as a finite intersection of "simpler" sets X_j, for which the projection is easily performed, and subsequent realization of cyclic projections for X_j. This idea has been well investigated for solving the so-called convex feasibility problems [4]. With the development of computer technology and the introduction of parallel computations this idea has been modified into the simultaneous projection of a given point onto several sets [5, 6].

Notice that there is an alternative approach for overcoming the problem of calculation of $\pi_X(\cdot)$. It consists in approximation at a current point the giving set X by super-half-space containing X and performing projection onto this space [7, 8].

In this paper we propose the sequential projections method with envelope stepsize control for solving variational inequality (1). This method is based on Fejer algorithms with an adaptive step [9], [10].

Assume that the set X can be decomposed as an intersection m super-sets X_j for which projection operation is easily performed, $X = \bigcap_{j \in J} X_j$, $J = \{1, 2, \ldots, m\}$. Let B is a unit ball, $\operatorname{conv} P$ is a convex hull of a set of vectors $P = \{p^i\}_{i \in I}$, I is a finite index set. The sequential projections method can be described a following iterative scheme.

Algorithm 1

Step 0 Give x^0 arbitrarily. Choose numbers $\lambda_0 > 0$, $\mu_0 > 0$, q_1, q_2, $\theta_0 \in (0, 1)$. Let $P^0 = \emptyset$. Set $k = 0$.

Step 1 Find $j \in J$ such that

$$\bar{x}(\lambda_k) = x^k - \lambda_k G(x_k) \notin X_j \qquad (3)$$

and calculate a direction

$$p^k = (\pi_{X_j}(\bar{x}(\lambda_k)) - x^k)/\mu_k, \quad P^k = P^k \cup \{p^k\}.$$

Step 2 Determine

$$x^{k+1} = x^k + d(x^k, \lambda_k, \mu_k), \quad d(x^k, \lambda_k, \mu_k) = \mu_k p^k.$$

Step 3 If $0 \in \operatorname{conv} P^k + \theta_k B$ then $\mu_{k+1} = q_1 \mu_k$, $\theta_{k+1} = q_2 \theta_k$. Set $k = k + 1$ and go to Step 1.

It can be shown that the Algorithm 1 converges to the set X^* under following assumptions:

– $(-G)$ is a strong locally restricted attractant of X^*, that is for each $\bar{x} \in X \backslash X^*$ there exists a neighborhood $U(\bar{x}, \varepsilon) = \{x : \|x - \bar{x}\| \leq 0\}$ such that

$$G(x^*)(x - x^*) \geq \delta, \quad x \in U(\bar{x}, \varepsilon) \cap X, \ x^* \in X^*,$$

for some $\delta > 0$;

– stepsize condition for λ_k: $\lambda_k \to +0$ when $k \to \infty$ and $\sum_{k=0}^{\infty} \lambda_k = \infty$;

– stationary condition $\theta_k \to +0$ when $k \to \infty$.

The Algorithm 1 may be modified for multiprocessor computers with using parallel calculations. It is possible if on the Step 1 we pick out several sets X_j for which (3) holds. Directions p^k are computed simultaneously by different processors for each $j \in J_k = \{j \in J : \bar{x}^k(\lambda_k) \notin X_j\}$. As result we obtain a collection of points $(x^{k+1})^j$, $j \in J_k$. Next point is determined as $x^{k+1} = \sum_{j \in J_k} w_j(x^{k+1})^j$, where $w_j > 0$ are such that $\sum_{j \in J_k} w_j = 1$.

In the report the numerical results and conclusions from the comparison proposed algorithms with previously developed will also be given.

References

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