# Convergence rate of Pareto frontier approximation in nonconvex multicriteria problems

A. I. Pospelov*

*DATADVANCE LLC, IITP RAS, alexis.pospelov@datadvance.net

## 1 Introduction

The mathematical simulation is a conventional tool for search effective solutions of complex problems. In many real-life applications the necessity to take into account different contradictory requirements leads to formulating investigated problems in a form of multicriteria optimization problems. In this work we consider the following multicriteria problem

$$\begin{aligned} &f(x) \to \min \\ &x \in X \subset \mathbb{R}^n \\ &f : \mathbb{R}^n \to \mathbb{R}^m \end{aligned} \tag{1}$$

Here $\mathbb{R}^n$ is a decision space, $X$ is a set of admissible decisions, $\mathbb{R}^m$ is a criterion space, and $f$ is a vector-function that images decisions vector into criterion one. Let $Y = f(X)$ is the image of all admissible decisions in the criterion space.

Important role in mulitcritrea optimization are played by an optimality notion. There are a number of ways to define optimality in multictiteria optimization [1]. In this work we consider Pareto optimality.

Here is a mathematical formulation of Pareto optimality.

**Definition 1** *Let $x'$ and $x''$ are two decisions from $X$. The decision $x'$ is said to Pareto dominate the decsion $x''$ according to given set of criteria $f(\cdot) = (f_1(\cdot), f_2(\cdot), \ldots, f_m(\cdot))$, if $f_i(x') \leq f_i(x'')$, $i = 1, 2, \ldots, m$, and $f(x') \neq f(x'')$.*

**Definition 2** *In multicriteria (1) the solution $f(x')$ is called **Pareto optimal** and its corresponding decision $x' \in X$ is called **Pareto efficient**, if there is no decision $x'' \in X$, that Parto dominates $x'$.*

The set of all Pareto optimal solutions is called Pareto frontier. We denote that as $P(Y)$.

We consider the problem of approximating the Pareto frontier. The approximation of the Pareto frontier is a classical problem in operations research and decision making support (e.g. see [2, 3]) and is of considerable applied importance, because information on the Pareto frontier is used in effective decision support systems involving multiple criteria. In particular, the approximation of the Pareto frontier is a central stage in the feasible goals method, in which the choice of a target point is based on computer visualization of the multidimensional Pareto frontier (see [2]). Due to this visualization, a decision maker can study in a visual form the possible criteria values and substitutions of criteria, which should help him or her to choose the best solution.

In this work we construct an approximation of the Pareto frontier as sequence of sets $T_i$ such as

$$\lim \delta^H(T_l, P(Y)) = 0,$$

where $\delta^H$ is Hausdorff distance,

$$\delta^H(C_1, C_2) = \\ = \max\left\{ \sup_{y \in C_2} \inf_{x \in C_1} \|x - y\|, \sup_{x \in C_1} \inf_{y \in C_2} \|x - y\| \right\}.$$

Depending on features of criteria functions $f$ and structure of $X$ different approaches can be involved in approximation of Pareto frontier. For instance in the convex case for approximating of

$P(Y)$ may be used method based on linear convolution of criteria. However such methods implicitly uses the fact that convex body can be represented by its support function. In the nonconvex case stochastic approaches or methods used convolution based on Chebyshev distance can be involved in approximation [6, 7, 4, 5].

It's known that using Chebyshev distance with different parameters it is possible to approximate nonconvex Pareto frontier with arbitrary accuracy. But there is still a question how to choose parameters to build approximation efficiently. In this work our goal is to estimate convergence rate of approximation of the Pareto frontier for algoritm that chooses parameters of convolution adaptively.

Firstly, we describe the algorithm. Similar algorithms were described early in [7, 9], but seems there are some differences in details that are important in our analysis.

## 2 Algorithm

Before describing the algorithm we need to introduce some additional notations. For given an axis-parallel box $B$ let $V^L(B)$ is its vertex that has maximal values of coordinates and $V^C(B)$ are vertexes that share an edge with $V^L(B)$.

The idea of algorithms is to build new $T_{i+1}$ a set of points each of which is result of solving a auxiliary single criteria optimization problem constructed using information from $T_i$.

Let on the initial iteration $T_0 = \{p^1, p^2, \dots p^m\}$, where $p^i$ is optimal value for optimization problems

$$f_i(x) \to \min,$$
$$x \in X.$$

Such points define an unique axis-parallel box $B_0$ in $\mathbb{R}^m$. Let $\mathcal{B}_0 = \{B_0\}$. Let initial approximation $T_0 = V^L(B_0)$.

Before each next step $l$ we have the set of boxes $\mathcal{B}_{l-1}$ and the approximation

$$T_{l-1} = \bigcup_{B \in \mathcal{B}} V^C(B).$$

On an arbitrary step $l$ for each $B_i \in \mathcal{B}_{l-1}$ find $p^i$ by solving the auxiliary problems

$$\max_i (f_i(x)) \to \min,$$
$$x \in X.$$

Choose $B_{i_0}$ such $||p^{i_0} - V^L(B_{i_0})||_\infty$ is maximal among other $p^i$. Let new approximation $T_l = T_{l-1} \cup p^{i_0}$. Let new set of boxes

$$\mathcal{B}_l = \mathcal{B}_{l-1} \cup Q(V^C(B_{i_0}), p^{i_0}) \backslash B_{i_0},$$

where $Q(p^1, p^2, \dots p^{n+1})$ are set of all possible axis-parallel boxes $B$ such $\text{card}(Q(p^1, p^2, \dots p^{n+1}) \backslash V^C(B)) = 1$.

The illustration shows approximation process in 2D.

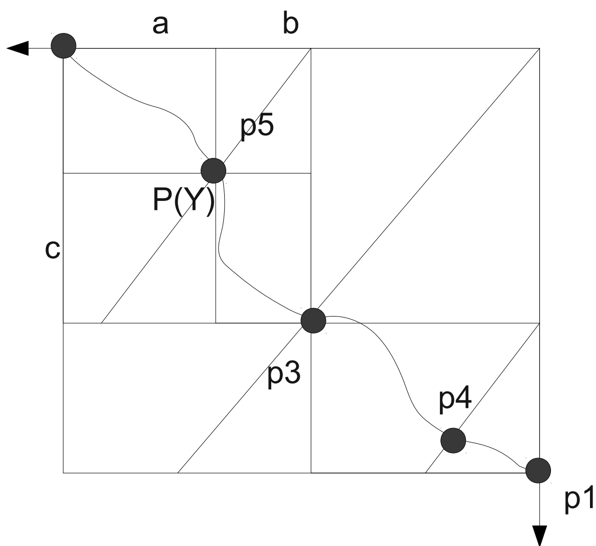


Figure 1: Schema of approximation

It is evident that on each step accuracy $\delta^H(T_l, P(Y))$ can be estimate by $||p^{i_0} - V^L(B_{i_0})||_\infty$. In the next section we formulate convergence rate for the algorithm.

## 3 Convergence rate

**Theorem 1** *Let $T_l$ is an approximation built by the algorithm for Pareto frontier $P(Y)$ in multicriteria problem* (1). *And let*

$$\delta^H(T_l, P(Y)) \le \varepsilon.$$

*Let $T^*$ is any other approximation of $P(Y)$ such*

$$\delta^H(T^*, P(Y)) \leq \varepsilon.$$

*Then*

$$\frac{\text{card}T_l}{\text{card}T^*} \leq 2^m$$

The idea of the proof is based on the fact that distance between new approximation and Pareto frontier can be estimate by distance between subsequent approximation, moreover

$$\delta^H(T_l, P(Y)) \leq \delta^H(T_{l+1}, T_l)$$

That means that on the each step we add to the approximation so called deep hole [10], which allow to estimate the convergence rate respect to the best possible algortim.

## References


[1] R. E. Steuer, *Multiple Criteria Optimization: Theory, Computation and Application.* John Wiley, New York, 1986.

[2] A. V. Lotov, V. A. Bushenkov, and G. K. Kamenev, *Interactive Solution Maps: Approximation and Visualization of Pareto Frontier.* Kluwer, Boston, 2004.

[3] K. Miettinen, *N*onlinear Multiobjective Optimization. Kluwer, Boston, 1999.

[4] K. Deb, *Multi-Objective Optimization Using Evolutionary Algorithms.* Wiley, Chichester, 2001.

[5] V. E. Berezkin, G. K. Kamenev and A. V. Lotov, *Hybrid Adaptive Methods for Approximating a Nonconvex Multidimensional Pareto Frontier* // Comput. Maths. Math. Phys., 2006. Vol. 46. N11. Pp. 1918-1931.

[6] N. M. Popov, *On the Approximation of the Pareto Set by the Convolution Method.* Vestn. Mosk. Gos. Univ. Vychisl. Mat. Kibern., No. 2, Pp. 3541, 1982.

[7] K. Klamroth, J. Tind and M. Wiecek, *Unbiased Approximation in Multicriteria Optimization.* 2002

[8] A. V. Lotov and I. I. Pospelova, *Lectures on the Theory and Methods of Multicriteria Optimization.* Mosk. Gos. Univ., Moscow, 2006 [in Russian].

[9] V. V. Morozov, *On approximation Pareto set with given accuracy in multicriteria problems.* Sistemy: matematicheskie metody opisanija, SAPR i upravlenija. - Kalinin: KGU, 1989, Pp 117-126, [in Russian].

[10] Kamenev G.K. Approximation of Completely Bounded Sets by the Deep Holes Method / / Zh. Vychisl. Matem. Matem. Fiz. 2001. Vol. 41. N11.