Bilevel problems with matrix and bimatrix games at the lower level and d.c. constraint optimization

A. V. Orlov*

*Institute for System Dynamics and Control Theory SB RAS, anor@icc.ru

A hierarchy is one of the promising paradigm in mathematical programming in recent years [1]. The pioneering work on bilevel optimization [2] led to the monograph on the class of mathematical programs with equilibrium constraints (MPECs) [3] and, somewhat later, to the monographs on the bilevel programming problems (BPPs) [4], [5].

There are a lot of applications of the MPECs and BPPs in control, economy, traffic, telecommunication networks, etc. (see, e.g., [6]). An investigation of MPECs or BPPs in the view of elaboration of the efficiency numerical methods is the urgent challenge of contemporary theory and methods of Mathematical Optimization [1].

One of most interesting classes of bilevel problems is the class with several players at the lower level. Also, we suppose, that players at the lower level depend on each other and we should find some equilibrium for these players. Such a formulation of the bilevel problem provides an interesting link between the hierarchy and competition. The latter is another promising paradigm according to Pang [1]. In this work we investigate two classes of BPPs with the equilibrium at the lower level.

The bilevel problems in the classical statement [4], [5] represent optimization problems, which – side by side with ordinary constraints such as equalities and inequalities – include a constraint described as an optimization subproblem:

$$\left.\begin{array}{c} F(x,y)\downarrow \min\limits_{x,y},\;\; x\in D,\;\; y\in Y_*(x),\\ Y_*(x)\stackrel{\triangle}{=}\operatorname*{Argmin}\limits_{y}\{G(x,y)\mid (x,y)\in D_1\}. \end{array}\right\} \qquad (\mathcal{BP})$$

Note, we consider an optimistic formulation of the bilevel problem (the goal of the upper level can be adjust with the actions of the lower level). Therefore the goal function F is to be minimized w.r.t. x and y simultaneously.

One or several players, which are subordinated to the upper level can be modeled by the optimization subproblem. If we model more than one player, we need to assume that all y_j, $j=1,...,n$ do not depend on each other. In that case the only "aggregative" player is operating at the lower level actually. On the one hand, such a model allows to investigate bilevel problems with the multiple players at the lower level. Note, the problems with several players at the lower level are very often arising from practice (e.g. the corporation may have several branches). On the another hand, the assumption of the independence of y_j can reduce the adequacy of the model.

In this work the generalized bilevel optimization problem is investigated. In the generalized bilevel problem the parametric game problem instead of parametric optimization subproblem is formulated at the lower level:

$$\left.\begin{array}{c} F(x,y_1,...,y_N)\uparrow \max\limits_{x,y_1,...,y_N},\\ x\in X,\\ (y_1,...,y_N)\in NE(\Gamma(x))(PE(\Gamma(x))), \end{array}\right\} \qquad (\mathcal{BP}_\Gamma)$$

where $NE(\Gamma(x))(PE(\Gamma(x)))$ is the set of Nash (Pareto) equilibrium points of the game

$$G_k(x,y_1,...,y_N)\uparrow \max_{y_k}, y\in Y_k(x), k=1,...,N. \qquad (\Gamma(x))$$

Here N is the number of players at the lower level. Each player at the lower level aims to maximize its profit functions G_k. And the player at the upper level aims to maximize profit function F subject to finding an equilibrium (Nash, Pareto) for players at the lower level. It seems that direct investigation of the problem $(\mathcal{BP}_{\Gamma})$ with the purpose of elaboration of solution methods is difficult to realize now. Therefore, we offer to investigate special classes of the problem $(\mathcal{BP}_{\Gamma})$.

Let us formulate the simplest case of such a problem with the parametric matrix game [7] at the lower level and linear goal functions with linear constraints at the upper level:

$$\left.\begin{array}{c} \langle c,x\rangle + \langle d_1,y\rangle + \langle d_2,z\rangle \uparrow \max\limits_{x,y,z}, \\ x \in X = \{x \in I\!R^m \mid Ax \le a,\ x \ge 0, \\ \langle b_1,x\rangle + \langle b_2,x\rangle = 1\},\ \ (y,z) \in C(\Gamma M(x)), \end{array}\right\} \qquad (\mathcal{BP}_{\Gamma M})$$

where $C(\Gamma(x))$ is the set of saddle points of the game

$$\left.\begin{array}{c} \langle y,Bz\rangle \uparrow \max\limits_{y}, \\ y \in Y(x) = \{y \mid y \ge 0, \langle e_{n_1},y\rangle = \langle b_1,x\rangle\}, \\ \langle y,Bz\rangle \downarrow \min\limits_{z}, \\ z \in Z(x) = \{z \mid z \ge 0, \langle e_{n_2},z\rangle = \langle b_2,x\rangle\}. \end{array}\right\} \qquad (\Gamma M(x))$$

$c, b_1, b_2 \in I\!R^m,\ y, d_1 \in I\!R^{n_1};\ z, d_2 \in I\!R^{n_2};\ a \in I\!R^p;$ $b_1 \ge 0,\ b_1 \ne 0,\ \ b_2 \ge 0,\ b_2 \ne 0;$ A, B are matrices of appropriate dimension; $e_{n_1} = (1, ..., 1)$, $e_{n_2} = (1, ..., 1)$ are vectors of appropriate dimension.

Also we investigate a bilevel problem with the simplest nonantagonistic conflict at the lower level, represented as a parametric bimatrix game [7] with matrices B_1 and B_2:

$$\left.\begin{array}{c} \langle c,x\rangle + \langle d_1,y\rangle + \langle d_2,z\rangle \uparrow \max\limits_{x,y,z}, \\ x \in X = \{x \in I\!R^m \mid Ax \le a,\ x \ge 0, \\ \langle b_1,x\rangle + \langle b_2,x\rangle = 1\}, \\ (y,z) \in NE(\Gamma B(x)), \end{array}\right\} \qquad (\mathcal{BP}_{\Gamma B})$$

where $NE(\Gamma(x))$ is the set of Nash equilibrium points of the game

$$\left.\begin{array}{c} \langle y,B_1z\rangle \uparrow \max\limits_{y}, \\ y \in Y(x) = \{y \mid y \ge 0, \langle e_{n_1},y\rangle = \langle b_1,x\rangle\}, \\ \langle y,B_2z\rangle \uparrow \max\limits_{z}, \\ z \in Z(x) = \{z \mid z \ge 0, \langle e_{n_2},z\rangle = \langle b_2,x\rangle\}. \end{array}\right\} \qquad (\Gamma B(x))$$

$c, b_1, b_2 \in I\!R^m,\ y, d_1 \in I\!R^{n_1};\ z, d_2 \in I\!R^{n_2};\ a \in I\!R^p;$ $b_1 \ge 0,\ b_1 \ne 0,\ \ b_2 \ge 0,\ b_2 \ne 0;$ A, B_1, B_2 are matrices of appropriate dimension; $e_{n_1} = (1, ..., 1)$, $e_{n_2} = (1, ..., 1)$ are vectors of appropriate dimension.

The expression $\langle b_1,x\rangle + \langle b_2,x\rangle = 1$ can be interpreted as some resource, which should be distributed by the leader among the followers.

In order to elaborate numerical methods for the solving of bilevel problems $(\mathcal{BP}_{\Gamma M})$ and $(\mathcal{BP}_{\Gamma B})$ we need to reformulate these problems as single level problems. Further to this end optimality conditions for generalized matrix game $(\Gamma M(x))$ and generalized bimatrix game $(\Gamma B(x))$ are considered.

Let us formulate a generalized matrix game with parameters ξ_1 and ξ_2:

$$\left.\begin{array}{c} \langle y,Bz\rangle \uparrow \max\limits_{y}, \\ y \in Y = \{y \mid y \ge 0, \langle e_{n_1},y\rangle = \xi_1 > 0\}, \\ \langle y,Bz\rangle \downarrow \min\limits_{z}, \\ z \in Z = \{z \mid z \ge 0, \langle e_{n_2},z\rangle = \xi_2 > 0\}. \end{array}\right\} \qquad (\Gamma M)$$

Recall, that the solution of the game (ΓM) with fixed ξ_1 and ξ_2 is defined as follows [7].

Definition 1 *The tuple (y^*, z^*) be called a saddle point of the game (ΓM) iff*

$$\left.\begin{array}{c} \forall y \in Y \quad \langle y,Bz^*\rangle \le v_* \stackrel{\triangle}{=} \\ \stackrel{\triangle}{=} \langle y^*,Bz^*\rangle \le \langle y^*,Bz\rangle \quad \forall z \in Z. \end{array}\right\} \qquad (1)$$

Here v_* is an optimal value of the game (ΓM).

Now, we can formulate optimality conditions for generalized matrix game (ΓM). These conditions are a generalization of classical optimality conditions in a matrix game [7].

Theorem 1 *The tuple $(y^*, z^*) \in C(\Gamma M)$ if and only if there exists a number v_* such that the following system is fulfilled:*

$$\left.\begin{array}{c} \xi_1(Bz^*) \le v_* e_{n_1}, \quad z^* \ge 0, \quad \langle e_{n_2},z\rangle = \xi_2; \\ \xi_2(y^*B) \ge v_* e_{n_2}, \quad y^* \ge 0 \quad \langle e_{n_1},y\rangle = \xi_1. \end{array}\right\} \qquad (2)$$

Note, conditions (2) represent finite numbers of equalities and inequalities.

Similarly, for generalized bimatrix game

$$\left.\begin{array}{c} \langle y, B_1 z\rangle \uparrow \max\limits_{y}, \\ y \in Y = \{y \mid y \geq 0, \langle e_{n_1}, y\rangle = \xi_1 > 0\}, \\ \langle y, B_2 z\rangle \uparrow \max\limits_{z}, \\ z \in Z = \{z \mid z \geq 0, \langle e_{n_2}, z\rangle = \xi_2 > 0\}, \end{array}\right\} \quad (\Gamma B)$$

we can define a solution as follows [7].

Definition 2 *The tuple (y^*, z^*) be called a Nash equilibrium point of the game (ΓB) iff*

$$\left.\begin{array}{c} \alpha_* \triangleq \langle y^*, B_1 z^*\rangle \geq \langle y, B_1 z^*\rangle \quad \forall y \in Y, \\ \beta_* \triangleq \langle y^*, B_2 z^*\rangle \geq \langle y^*, B_2 z\rangle \quad \forall z \in Z. \end{array}\right\} \quad (3)$$

Optimality conditions for generalized bimatrix game (ΓB) are a generalization of classical optimality conditions in a bimatrix game [7].

Theorem 2 *The tuple $(y^*, z^*) \in NE(\Gamma B)$ if and only if there exist numbers α_* and β_* such that the following system is fulfilled:*

$$\left.\begin{array}{c} \xi_1(B_1 z^*) \leq \alpha_* e_{n_1}, \;\; \xi_2(y^* B_2) \leq \beta_* e_{n_2}; \\ \langle y^*, (B_1 + B_2) z^*\rangle = \alpha_* + \beta_*; \\ y^* \geq 0, \; \langle e_{n_1}, y\rangle = \xi_1; \quad z^* \geq 0, \; \langle e_{n_2}, z\rangle = \xi_2. \end{array}\right\} \quad (4)$$

Here α_* and β_* are payoffs of the first and the second players in the game (ΓB) respectively.

Let us draw attention to the equality $\langle y^*, (B_1 + B_2) z^*\rangle = \alpha_* + \beta_*$ in (4). This equality creates basic complexity in system (4).

Now we can replace a game at the lower level by its optimality conditions. So, for the bilevel problem $(\mathcal{BP}_{\Gamma M})$ it is possible to formulate the following equivalent single level problem:

$$\left.\begin{array}{c} \langle c, x\rangle + \langle d_1, y\rangle + \langle d_2, z\rangle \uparrow \max\limits_{x,y,z,v}, \\ Ax \leq a, \;\; x \geq 0, \;\; \langle b_1, x\rangle + \langle b_2, x\rangle = 1, \\ y \geq 0, \;\; \langle e_{n_1}, y\rangle = \langle b_1, x\rangle, \\ z \geq 0, \;\; \langle e_{n_2}, z\rangle = \langle b_2, x\rangle, \\ \langle b_1, x\rangle (Bz) \leq v e_{n_1}, \\ \langle b_2, x\rangle (yB) \geq v e_{n_2}. \end{array}\right\} \quad (\mathcal{PM})$$

More precisely, the following theorem takes place.

Theorem 3 *The triplet (x^*, y^*, z^*) is a global optimistic solution of the bilevel problem $(\mathcal{BP}_{\Gamma M})$, if and only if there exist a number v_* such that the 4-tuple (x^*, y^*, z^*, v_*) is a global solution of problem $(\mathcal{PM})$.*

It can readily be seen, that problem $(\mathcal{PM})$ is a global optimization problem with a nonconvex feasible set (see, e.g., [8]–[10]). A nonconvexity in the problem $(\mathcal{PM})$ generated by a group of $(n_1 + n_2)$ bilinear constraints. These constraints arise from optimality conditions for the generalized matrix game at the lower level of the bilevel problem $(\mathcal{BP}_{\Gamma M})$. It is known, that bilinear function is represented as a difference of two convex functions (i.e. bilinear function is d.c. function) [11]–[12]. So, problem $(\mathcal{PM})$ belongs to the class of nonconvex optimization problems with d.c. constraints [9].

As far as bilevel problem with the game (ΓB) is concerned we obtain the following single level problem similarly:

$$\left.\begin{array}{c} \langle c, x\rangle + \langle d_1, y\rangle + \langle d_2, z\rangle \uparrow \max\limits_{x,y,z,\alpha,\beta}, \\ Ax \leq a, \;\; x \geq 0, \;\; \langle b_1, x\rangle + \langle b_2, x\rangle = 1, \\ y \geq 0, \;\; \langle e_{n_1}, y\rangle = \langle b_1, x\rangle, \\ z \geq 0, \;\; \langle e_{n_2}, z\rangle = \langle b_2, x\rangle, \\ \langle b_1, x\rangle (B_1 z) \leq \alpha e_{n_1}, \\ \langle b_2, x\rangle (y B_2) \geq \beta e_{n_2}, \\ \langle y, (B_1 + B_2) z\rangle = \alpha + \beta. \end{array}\right\} \quad (\mathcal{PB})$$

Also, the following theorem takes place.

Theorem 4 *The triplet (x^*, y^*, z^*) is a global optimistic solution of the bilevel problem $(\mathcal{BP}_{\Gamma B})$, if and only if there exist numbers α_* and β_* such that the 5-tuple $(x^*, y^*, z^*, \alpha_*, \beta_*)$ is a global solution of problem $(\mathcal{PB})$.*

The problem $(\mathcal{PB})$ is a global optimization problem with a nonconvex feasible set too. And a nonconvexity in the d.c. constraint problem $(\mathcal{PB})$ generated by a group of $(n_1 + n_2 + 1)$ bilinear constraints. Besides, each set of bilinear constrains in the problems $(\mathcal{PM})$ and $(\mathcal{PB})$ is bilinear w.r.t. its pair of variables.

For the purpose of solving the d.c. constraint problems formulated above, we intend to construct the algorithms based on the Global Search Theory in d.c. optmization problems elaborated in [9], [13]. The approach allows to build efficient methods for finding global solutions in various d.c. optimization problems [9], [12], [15]. Global Search Algorithms based on Global Search Theory consist of two principal stages:
1) a special local search methods, which takes into account the structure of the problem under scrutiny [9], [12], [14];
2) the procedures, based on Global Optimality Conditions [9], that allow to improve the point provided by the Local Search Method [9].

This work is carried out under financial support of Russian Foundation for Basic Research (project no. 11-01-00270-a).

References

[1] J.-S. Pang, *Three modelling paradigms in mathematical programming*, Mathematical programming, Ser.B., Vol.125, pp. 297-323, 2010.

[2] J. Bracken, J. T. McGill, *Mathematical programs with optimization problems in the constraints*, Operations research, Vol.21, pp. 37-44, 1973.

[3] Z.-Q. Luo, J.-S. Pang, D. Ralph *Mathematical programs with equilibrium constraints*, Cambridge: Cambridge University Press, 1996.

[4] J. F. Bard, *Practical Bilevel Optimization*, Dordrecht, The Netherlands: Kluwer Academic Publishers, 1998.

[5] S. Dempe, *Foundations of Bilevel Programming*, Dordrecht, The Netherlands: Kluwer Academic Publishers, 2002.

[6] B. Colson, P. Marcotte, G. Savard, *An overview of bilevel optimization*, Annals of operations research, Vol.153, pp. 235–256, 2007.

[7] G. Owen, *Game Theory*, UK: Academic Press, 1995.

[8] R. Horst, H. Tuy, *Global Optimization. Deterministic Approaches*, Berlin, Germany: Springer-Verlag, 1993.

[9] A. S. Strekalovsky, *Elements of nonconvex optimization*, Novosibirsk: Nauka, 2003 (in russian).

[10] R. G. Strongin, Ya. D. Sergeyev, *Global Optimization with Non-Convex Constraints. Sequential and Parallel Algorithms*, New York: Springer-Verlag, 2000.

[11] C. A. Floudas, V. Visweswaran, *Quadratic optimization*, In “Handbook of Global Optimization”, Ed. by Horst, R., Pardalos, P. Dordrecht, The Netherlands: Kluwer Academic Publishers, pp. 224–228, 1995.

[12] A. S. Strekalovsky, A. V. Orlov *Bimatrix games and bilinear programming*, Moscow: FizMatLit, 2007 (in russian).

[13] A. S. Strekalovsky *Minimizing sequences in problems with D.C. constraints*, Computational Mathematics and Mathematical Physics, Vol.45, No.3, pp.947–960, 2005.

[14] T. V. Gruzdeva, A. S. Strekalovsky, *Local search in problems with nonconvex constraints*, Computational Mathematics and Mathematical Physics, Vol.47, No.3, pp.381–396, 2007.

[15] A. S. Strekalovsky, A. V. Orlov, A. V. Malyshev, *On computational search for optimistic solutions in bilevel problems*, Journal of Global Optimization, Vol. 48, No. 1, pp.159-172, 2010.