On the Approach of Obtaining the Upper Bounds on the Average Number of Iterations of Some Integer Programming Algorithms

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1 Introduction

In the area of discrete optimization the problem of analyzing the behavior of algorithms on the average by solving various NP-hard problems is important. In particular it concerns finding polymomially solvable on the average classes of problems (see [1, 3, 6, 8]).

Some classes of the multi-dimensional knapsack problem with Boolean variables and set packing problem were described in [6], where the algorithm of dynamic programming (DP) is polynomial on the average, that is, the mathematical expectation of the operations number is bounded by the polynomial of the input length of the problem.

The approach to finding the upper bounds on the average number of iterations for some known algorithms of integer linear programming (ILP) using the continuous optimization was presented earlier in [9]. The idea of the approach lies in the usage of some determinated bounds on the number of algorithms iterations obtained by the regular partition method [4] and the upper bounds of the average cardinality of the feasible solutions set of the problem. The approach mentioned above was applied to the first Gomory cutting plane algorithm, the branch-and-bound method (the Land and Doig scheme) [7], L-class enumeration algorithm [4] for solving both the set packing problem and the multi-dimensional knapsack problem with Boolean variables.

In this paper we introduce the review of the results obtained earlier. Some new bounds for the set covering problem are presented as well.

2 Description of approach and review of results

Consider the problem of ILP:

$$\max\{\mathbf{cx} \mid \mathbf{Ax} \leq \mathbf{b},\ \mathbf{x} \geq \mathbf{0},\ \mathbf{x} \in Z^n\}. \quad (1)$$

Here $\mathbf{A} = ||a_{ij}||$ is the matrix of the order $m \times n$; $\mathbf{c} = (c_1, ..., c_n)$; $\mathbf{b} = (b_1, ..., b_m)$; $\mathbf{0}$ is the n-dimensional vector and $\mathbf{x} = (x_1, ..., x_n)^{\mathrm{T}}$ is the vector of the variables. The input data of the problem are integer-valued. Next suppose the set of the feasible solutions of the corresponding linear programming problem to be bounded. Denote by M the polyhedron of the problem.

Some results in the method of regular partitions are obtained on the basis of *L-partition* of the space R^n. The partition is defined as follows. Each point in Z^n forms the separate L-class, that is, the element of partition. The points $\mathbf{x},\ \mathbf{y} \in R^n$ ($\mathbf{x} \succ \mathbf{y}$ and $\mathbf{x},\ \mathbf{y} \notin Z^n$) belong to the same *fractional* L-class if no $\mathbf{z} \in Z^n$ exists such that $\mathbf{x} \succ \mathbf{z} \succ \mathbf{y}$. Here $\succ$ is the symbol of lexicographical comparison. Let $X \subset R^n$. The factor set X/L is called the *L-structure* of X, and its cardinality is denoted by $|X/L|$.

The subset $Q = \{V_s, V_{s+1}, ..., V_t\}$ of the fractional classes in X/L is called an *L-complex* if no $\mathbf{z} \in X \cap Z^n$ exists such that $\mathbf{x}^s \succ \mathbf{z} \succ \mathbf{x}^t$ for any $\mathbf{x}^s \in V_s,\ \mathbf{x}^t \in V_t$. Denote by $\Psi(X)$ the maximal cardinality of L-complexes induced by X.

Denote by $I_{\mathcal{A}}$ the number of iterations performed by the algorithm $\mathcal{A}$ for solving the ILP problem. Let us formulate some necessary determinate upper bounds on the iterations number for the being investigated algorithms. The bounds are obtained by the regular partition method [4].

Let f^* be the optimal value of the objective function of the problem (1), $\tilde{f}$ is the optimal value of the objective function of the corresponding linear programming problem. Denote by $D(n)$ the set of feasible solutions of the problem (1). For the first Gomory cutting plane algorithm (G1) it holds the following

$$I_{G1} \leq (\lfloor \tilde{f} \rfloor - f^* + 1)(|M/L| - |D(n)| + 1).$$

For the iterations number for the L-class enumeration algorithm (LCE) the upper bound is as follows

$$I_{LCE} \leq |M/L|.$$

The iteration here means the transition to the next L-class.

Using the definition of the $\Psi(X)$ it is not difficult to prove the inequality

$$|M/L| \leq |D(n)| + (|D(n)| + 1)\Psi(M).$$

Therefore, if it holds the polynomial upper bounds on maximal cardinality of L-complexes for M, the average number of iterations for the algorithms G1 and LCE may be obtained by means of the upper bounds on the average number of feasible solutions of the problem.

Let the matrix $\mathbf{A}$ and the vector $\mathbf{b}$ in the problem (1) be nonnegative. The number of solvable linear programming problems will further be considered as the number of iterations for the Land and Doig (LD) algorithm. Using the *lower cubic partition* of the space R^n it was estimated (see [4, 10]), that

$$I_{LD} \leq (2n + 1)|D(n)|.$$

Note that the analogous bounds for the algorithms hold for the ILP problem of the form:

$$\min\{\mathbf{cx} \mid \mathbf{Ax} \geq \mathbf{b},\ \mathbf{x} \geq \mathbf{0},\ \mathbf{x} \in Z^n\}.$$

Now we present the bounds of the average number of iterations for both the set packing problem and the multi-dimensional knapsack problem. The bounds are obtained within the approach described above along with the bounds on the value $|D(n)|$ on the average in [6].

The model of ILP for the set packing problem (SPP) looks as follows:

$$\max\{\mathbf{cx} | \mathbf{Ax} \leq \mathbf{e},\ \mathbf{x} \in \{0,1\}^n\},$$

where $\mathbf{A}$ is the Boolean matrix of the order $m \times n$ and $\mathbf{e} = (1, ..., 1)$ is the m-dimensional vector. Here and further $\mathbf{c} > \mathbf{0}$. Let us consider the class $\mathcal{P}(n, p)$ of the SPP, where all the elements of the matrix $\mathbf{A}$ are independent random variables, notably

$$P\{a_{ij} = 1\} = p,\ P\{a_{ij} = 0\} = 1 - p, \qquad (2)$$

where $p \in (0,\ 1)$. If the parameters of the problem satisfy the condition $mp^2 \geq \ln n$, then it holds the bound [6]:

$$\mathbf{E}|D(n, p)| \leq 2n + 1. \qquad (3)$$

Here and further $\mathbf{E}|D(n, p)|$ is the mathematical expectation of the cardinality of $D(n, p)$ for the ILP problems in the being investigated classes.

It is known that it holds $\Psi(M) = 1$ for the set packing problem [4]. Hence, taking into account the above property and some other factors of the problem polyhedron in [5] on the basis of (3) the following theorems are proved

Theorem 1 *For the algorithm G1 and unweighted problems from $\mathcal{P}(n, p)$ the estimate*

$$\mathbf{E}I_{G1} \leq 3n^2 - 2n - 1$$

takes place.

Theorem 2 *For the problems from $\mathcal{P}(n, p)$ and the algorithm LCE with the initial value of the objective function equal to* $\max\{c_j | j = 1, ..., n\}$ *it holds the bound*

$$\mathbf{E}I_{LCE} \leq 3n + 1.$$

Now consider the multi-dimensional knapsack problem, where all the variables are Boolean and $\mathbf{A} \geq \mathbf{0}$, $\mathbf{b} > \mathbf{0}$, contrary to (1). Let $b_i \leq B$, $i = 1, ..., m$, where $B > 0$. All the elements of the matrix $\mathbf{A}$ are the independent random variables, where $P\{a_{ij} = t\} = p$ for $t = 1, ..., B$ and $P\{a_{ij} = 0\} = 1 - pB > 0$. The parameters of the problem satisfy the condition $m(pB)^2 \geq \gamma \ln n$, where the constant $\gamma \geq 13$. For the average value of $|D(n,p)|$ of such problem it holds the bound (3) as well [6].

Consider the knapsack problem class $\mathcal{K}(n,p)$, which includes also the case with the uniform discrete distribution of the elements of the matrix $\mathbf{A}$ in the set $\{1, ..., B\}$. The bound (3) takes place for the $\mathcal{K}(n,p)$ [11].

It is known that $\Psi(M) \leq n$ for the knapsack problem (see [4]). Using this and other properties of the problem the upper bounds on the average number of iterations for the considered algorithms are obtained in [11]. For example, it holds

Theorem 3 *For the problems from $\mathcal{P}(n,p)$ and $\mathcal{K}(n,p)$ the inequality*

$$\mathbf{E}I_{LD} \leq 4n^2 + 4n + 1$$

takes place.

In the case of the weighted problems of both types the algorithm G1 becomes pseudopolynomial. In [9] the revised version of this algorithm is developed, having the polynomial upper bound on the average number of iterations for the problems from $\mathcal{K}(n,p)$ and $\mathcal{P}(n,p)$ at random $c_j,\ j = 1, ..., n$.

3 New polynomially solvable cases for SPP and SCP

In analyzing the problems with random input data the finding of classes, where $\mathbf{E}|D(n,p)| = O(n^k)$ is of interest. Let us remind that for $\mathcal{P}(n,p)$ and $\mathcal{K}(n,p)$ it holds $\mathbf{E}|D(n,p)| = O(n)$.

For the SPP problem the upper bounds on the average number of feasible solutions are obtained in [2]. Let us define the function for the fixed n, p

$$u(k) = \frac{\ln \frac{n-k}{k+1}}{\ln \frac{1+(k-1)p}{(1-p)(1+kp)}}, \quad k = 1, ..., n.$$

Theorem 4 *For every integer $k \geq 1$ and SPP with $n \geq k+1$, $m \geq u(k)$ it holds the inequality*

$$\mathbf{E}|D(n,p)| \leq \frac{n^k - 1}{n - 1} + n^{k+1}. \qquad (4)$$

Above all, under certain relations between the problem parameters the lower bounds for $\mathbf{E}|D(n,p)|$ are obtained in [2]. On the basis of these bounds the new class of SPP problems is yielded, $q \leq \mathbf{E}|D(n,p)| \leq \tilde{q}$, where $q = O(n^2)$ and $\tilde{q} = O(n^3)$.

We have also investigated the set covering problem (SCP):

$$\min\{\mathbf{cx} | \mathbf{Ax} \geq \mathbf{e},\ \mathbf{x} \in \{0,1\}^n\}.$$

Let the elements of the Boolean matrix of $\mathbf{A}$ be the independent random variables satisfying the distribution (2).

Collaboratively with Gofman N.G. we got the analogous upper bounds for the SCP.

In agreement with the described approach from theorem 4 with fixed k we obtain the polynomial upper bounds on the average number of iterations for the Land and Doig algorithm, L-classes enumeration algorithm and for the unweighted problems for the first Gomory cutting plane algorithm. In addition, the classes of problems described in theorem 4 prove to be polynomially solvable on the average by the DP algorithms.

It is worth mentioning that the bound (4) is valid enough to describe the new polynomially solvable classes of SPP and SCP on average in spite of being quite the high one.

In the present report the results of the experimentation for the algorithms and problems under review are also given including the results obtained by means of IBM ILOG CPLEX.

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