# A decomposition method for approximating Pareto frontier in Multiobjective Integer Linear Problems

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## Introduction

Decision problems with multiple criteria and integer or mixed integer variables often arise in different areas such as forest planning and management, water resources management, telecommunication network planning, transportation and investment problems, etc ([1],[2],[3]).

Consider the following problem:

$$\max \ f_1(\boldsymbol{x}) = \boldsymbol{c}^1 \boldsymbol{x}$$
$$...$$
$$\max \ f_m(\boldsymbol{x}) = \boldsymbol{c}^m \boldsymbol{x} \qquad \text{(MOILP)}$$
$$\text{subject to}$$
$$\boldsymbol{x} \in X = \{\, \boldsymbol{x} \mid A\boldsymbol{x} \le \boldsymbol{b}, \ \boldsymbol{x} \ \text{integer}\}$$

where $\boldsymbol{x}$ is an $n$-dimensional vector of variables, $A$ is an $m \times n$ matrix, $\boldsymbol{b}$ is the RHS vector and the vectors $\boldsymbol{c}^i$ $(i = 1, ..., m)$ represent the coefficients of the objective functions (criteria).

Let's denote $y_i = f_i(\boldsymbol{x})$, $i = 1, ..., m$, and let $\boldsymbol{y} = (y_1, ..., y_m)$ be a vector in the criteria space. The set $Y \subset \mathbb{R}^m$ composed by all possible criterion vectors $\boldsymbol{y} = f(\boldsymbol{x})$ when $\boldsymbol{x} \in X$, is known as *Feasible Criteria Set* (FCS).

A point $\boldsymbol{y}' \in Y$ is *non-dominated* or *Pareto optimal* (*efficient*) if and only if there is no $\boldsymbol{y}'' \in Y$, $\boldsymbol{y}'' \neq \boldsymbol{y}'$, such that $y_j'' \ge y_j'$, $j = 1, ..., m$. The set $\mathrm{P}(Y)$ of all non-dominated points $\boldsymbol{y} \in Y$ is called *Pareto frontier*.

A *dominance cone* for a point $\boldsymbol{v} \in \mathbb{R}^m$ is defined as $\mathrm{D}(\boldsymbol{v}) = \{\boldsymbol{z} \in \mathbb{R}^m : \boldsymbol{z} = \boldsymbol{v} - \boldsymbol{w}, \ \boldsymbol{w} \in \mathbb{R}_+^m\}$, where $\mathbb{R}_+^m$ is the non-negative cone in $\mathbb{R}^m$.

The information about the Pareto frontier can be very useful for Decision Makers (DM) in Decision Support Systems (DSS) based on multiobjective models. Some examples of such DSSs are given in [4].

When we are only interested in Pareto frontier it is worth considering the *Edgeworth-Pareto Hull* (EPH) of $Y$ defined as $\mathrm{EPH}(Y) = \{\boldsymbol{z} \in \mathbb{R}^m : \boldsymbol{z} = \boldsymbol{y} - \boldsymbol{w}, \ \boldsymbol{y} \in Y, \ \boldsymbol{w} \in \mathbb{R}_+^m\}$. The set $\mathrm{EPH}(Y)$ has the same Pareto frontier that $Y$ but its structure is simpler and therefore $\mathrm{EPH}(Y)$ is easier for constructing and visualizing.

Some effective algorithms for constructing and approximating EPH of convex sets are described in chapter 6 in [4]. These algorithms can be applied also for constructing and approximating EPHs of non-convex sets. It is important to remark that it is possible to know the vertices and hyperplanes corresponding to each facet of $\mathrm{EPH}(Y)$ since these algorithms describe the EPH of a set, simultaneously, as a list of vertices and as a system of linear inequalities.

When multiobjective programming problems have integer variables, even if the restrictions are linear, the set $Y$ can be non-convex. Unsupported efficient solutions of MOILP, i.e. solutions that do not belong to the frontier of the convex hull of the feasible region, cannot be obtained by optimizing scalar surrogate functions consisting in weighted-sums of the objective functions (*Weighted Objective Function Method*). The other traditional techniques, such as *methods based on the Tchebycheff Metric*, *epsilon-Constraining Method*, and some others, have difficulties in the case of MOILP with more than three objective functions [3].

The *Convex Edgeworth-Pareto Hull* (CEPH) of

a non-convex set $U$ is defined as $\mathrm{EPH}(conv(U))$, where $conv(U)$ denotes the convex hull of $U$. To calculate CEPHs, the algorithms described in [4] can be successfully applied. Here we describe a new iterative method that approximates the Pareto frontier in MOILP by pairs of interior and exterior estimates which converge each other during the iterations. These estimates are obtained by combining CEPHs constructed for a number of subsets $Y_k$ into which the original $Y$ is decomposed.

## Decomposition method based on CEPHs

We approximate $\mathrm{P}(Y)$ using a pair of sets $(I^{(k)}, E^{(k)})$ such that $I^{(k)} \subseteq \mathrm{EPH}(Y) \subseteq E^{(k)}$ and $d(I^{(k)}, E^{(k)}) \to 0$, $k \to \infty$, where $d(.,.)$ means a distance between two sets. Let's illustrate the main ideas of the method using a small bicriteria example. Figure 1 displays a set $Y$ in the criteria space $\{y_1, y_2\}$. When maximizing $y_1$ and $y_2$, the Pareto frontier $\mathrm{P}(Y)$ has eight non-dominaded points $\{P_1, ..., P_8\}$.
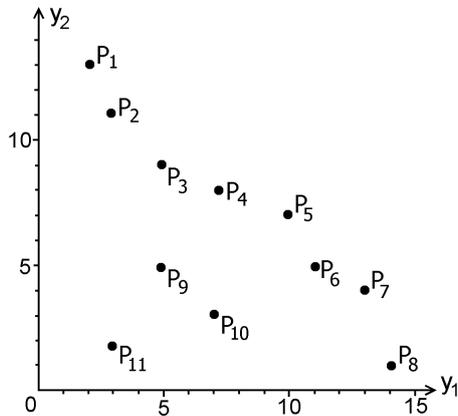


Figure 1

In the **initial iteration**, the CEPH($Y$) is calculated using the algorithms described in chapter 6 of [4]. These algorithms provide the double description of the CEPH – as a system of linear inequalities and as full list of its vertices. All vertices of CEPH($Y$) (*supported points*) form the initial list of the found Pareto frontier

points – $PFP^{(0)}$. In our example, $PFP^{(0)} = \{P_1, P_5, P_7, P_8\}$. Now we assume $E^0 = \mathrm{CEPH}(Y)$, and $I^0 = \bigcup_{\forall v^i \in PFP^{(0)}} \mathrm{D}(v^i)$, i.e. $I^0$ is the union of the dominance cones constructed for all supported points of $\mathrm{P}(Y)$.
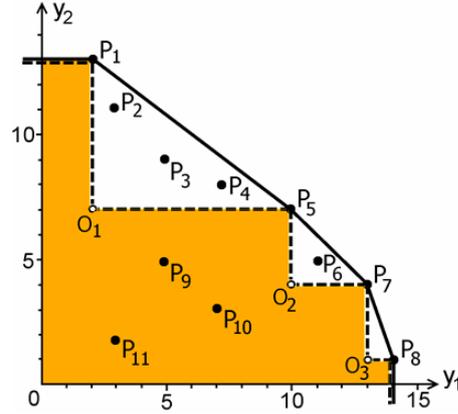


Figure 2

In Figure 2, the boundary of $E^0$ is given by solid line, and the boundary of $I^0$ is given by dashed line (the interior of $I^0$ is colored). It is obvious that $I^{(0)} \subseteq \mathrm{EPH}(Y) \subseteq E^0$, and all unsupported points of $\mathrm{P}(Y)$, not found yet, ($P_2, P_3, P_4$ and $P_6$) belong to the difference $E^{(0)} \backslash I^{(0)}$, i.e. they are located between solid and dashed lines in Figure 2. The set $E^{(0)} \backslash I^{(0)}$ can be divided into a number of not intersected subsets – "search zones" corresponding to the facets of CEPH($Y$). In our example, we have 3 search zones – triangles $P_1 O_1 P_5$, $P_5 O_2 P_7$ and $P_7 O_3 P_8$. The search zones are denoted as $\mathrm{SZ}(\boldsymbol{v}_1, ..., \boldsymbol{v}_s)$ where $(\boldsymbol{v}_1, ..., \boldsymbol{v}_s)$ is the list of vertices which belong to the corresponding facet, for example $\mathrm{SZ}(P_5, P_7)$.

Now, the original problem is decomposed into a number of the problems of finding the Pareto frontier points in relatively small disjoint search zones. At each iteration of the algorithm, we choose one of such zones and try to find new Pareto frontier points located on it. Finding new points, we change the boundaries of the interior and exterior estimates, $I^{(k)}$ and $E^{(k)}$, and, probably, create new, smaller in size, search zones.

Let's consider the **k-th iteration** in more detail. Suppose we have $I^{(k-1)}$ and $E^{(k-1)}$ estimates, and we want to analyze the search zone

$\mathrm{SZ}(\boldsymbol{v}_1,...,\boldsymbol{v}_s)$. The $\mathrm{SZ}(\boldsymbol{v}_1,...,\boldsymbol{v}_s)$ can be described as $\mathrm{SZ}(\boldsymbol{v}_1,...,\boldsymbol{v}_s) = \mathrm{D}(\boldsymbol{v}_1,...,\boldsymbol{v}_s) \setminus I^{(k-1)}$ where $\mathrm{D}(\boldsymbol{v}_1,...,\boldsymbol{v}_s)$ denotes the variety of dominated points for the facet defined by $(\boldsymbol{v}_1,...,\boldsymbol{v}_s)$:

$$\mathrm{D}(\boldsymbol{v}_1,...,\boldsymbol{v}_s) = \begin{cases} \boldsymbol{z} \in \mathbb{R}^m : \\ \boldsymbol{z} = \boldsymbol{u} - \boldsymbol{w}, \ \boldsymbol{w} \in \mathbb{R}_+^m \\ \boldsymbol{u} = \lambda_1 \boldsymbol{v}_1 + ... + \lambda_s \boldsymbol{v}_s \\ \lambda_1 + ... + \lambda_s = 1, \\ 0 \le \lambda_i \le 1, \ i = 1,...,s. \end{cases}$$

The geometric illustration of $\mathrm{D}(P_1, P_5)$ is given in Figure 3a and the corresponding $\mathrm{SZ}(P_1, P_5)$ is shown in Figure 3b.
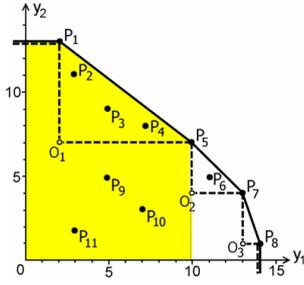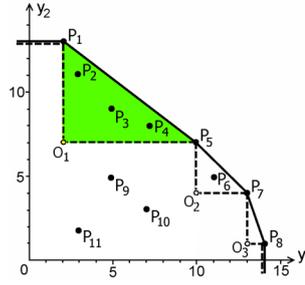


Figure 3a                        Figure 3b

To find new Pareto points, we consider the subvariety $Y^{(k)} = Y \cap \mathrm{SZ}(\boldsymbol{v}_1,...,\boldsymbol{v}_s) \setminus \mathrm{D}(\boldsymbol{v}_1) \setminus ... \setminus \mathrm{D}(\boldsymbol{v}_s)$ and construct $\mathrm{CEPH}(Y^{(k)})$ defined as follows:

$$\mathrm{CEPH}(Y^{(k)}) = \begin{cases} \boldsymbol{z} \in \mathbb{R}^m : \\ \boldsymbol{z} = \boldsymbol{y} - \boldsymbol{w}', \ \boldsymbol{w}' \in \mathbb{R}_+^m \\ \boldsymbol{y} = f(\boldsymbol{x}), \ \boldsymbol{x} \in X \\ \boldsymbol{z} = \boldsymbol{u} - \boldsymbol{w}'', \ \boldsymbol{w}'' \in \mathbb{R}_+^m \\ \boldsymbol{u} = \lambda_1 \boldsymbol{v}_1 + ... + \lambda_s \boldsymbol{v}_s \\ \lambda_1 + ... + \lambda_s = 1, \\ 0 \le \lambda_i < 1, \ i = 1,...,s, \\ \boldsymbol{z} \notin I^{(k-1)} \end{cases}$$

Assuming that $\mathrm{SZ}(P_1, P_5)$ was selected, the resulting $\mathrm{CEPH}(Y^{(k)})$ is presented in Figure 4. Its vertices $P_2$ and $P_4$ give two new points of $\mathrm{P}(Y)$.

It is easy to see that all the points of $P(Y)$ situated in $\mathrm{SZ}(P_1, P_5)$, and still unknown (in our example $- P_3$), belong to the colored area, i.e. $\mathrm{CEPH}(Y^{(k)}) \setminus I^{(k)}$, and none of these can belong to the dominance cones of the vertices of $\mathrm{CEPH}(Y^{(k)})$, i.e. $\mathrm{D}(P_2)$ and $\mathrm{D}(P_4)$. Using this observation, we can refine the interior and exterior estimates $I^{(k-1)}$ and $E^{(k-1)}$ and get a more

detail description of the Pareto frontier $\mathrm{P}(Y)$ (see Figure 5).
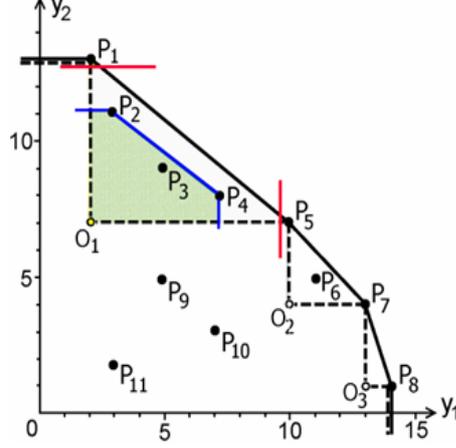


Figure 4
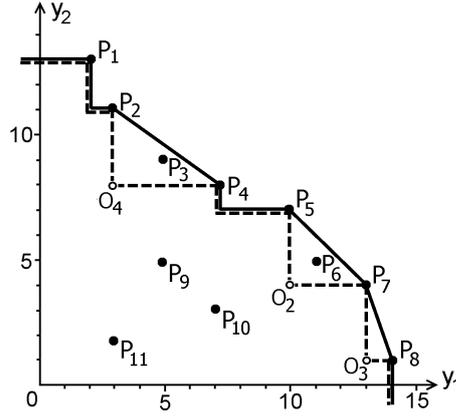


Figure 5

More precisely, 1) the vertices $\boldsymbol{v}_1^*,...,\boldsymbol{v}_q^*$ of the constructed $\mathrm{CEPH}(Y^{(k)})$ are included into the list of the Pareto frontier points: $PFP^{(k)} = PFP^{(k-1)} \cup \{\boldsymbol{v}_1^*,...,\boldsymbol{v}_q^*\}$; 2) $I^{(k)} = I^{(k-1)} \cup \mathrm{D}(\boldsymbol{v}_1^*) \cup ... \cup \mathrm{D}(\boldsymbol{v}_q^*)$; and 3) $E^{(k)} = E^{(k-1)} \setminus \mathrm{SZ}(\boldsymbol{v}_1,...,\boldsymbol{v}_s) \cup \mathrm{CEPH}(Y^{(k)})$.

For each facet of $\mathrm{CEPH}(Y^{(k)})$, a new search zone is created. In our example, it is $\mathrm{SZ}(P_2, P_4)$, i.e. the triangle $P_2 O_4 P_4$.

Here the $k$-th iteration terminates.$\square$

Repeating the iteration for search zones $\mathrm{SZ}(P_2, P_4)$ and $\mathrm{SZ}(P_5, P_7)$, we find the points $P_3$ and $P_6$. The analysis of $\mathrm{SZ}(P_7, P_8)$ does not give any new point. Now we have the situation shown

in Figure 6. The interior estimate coincides with the exterior estimate. This means that the list of the search regions is empty and all the points of $P(Y)$ have been found already.
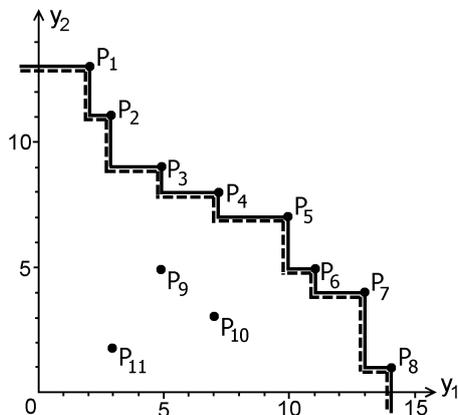


Figure 6

In practice, it does not necessarily to continue the construction of the Pareto frontier until the end. In many practical applications, its approximate description is sufficient.

To select a search zone for the next iteration, the different strategies can be used. For example, it is possible to choose the search zone with largest size or to take into account the preferences of the DM.

Now we give a formal description of the proposed algorithm.

### Algorithm

**Step 1**:
- construct $CEPH^{(0)} = CEPH(Y)$
- include all vertices of $CEPH(Y)$ into $PFP^{(0)}$
- set $I^{(0)} = \bigcup_{(\forall v^i \in PFP^{(0)})} D(v^i)$
- set $E^{(0)} = CEPH(Y)$
- set $k = 0$, $SZ\_List = \emptyset$.

**Step 2:**
- create search zones for all facets of $CEPH^{(k)}$ and include them into $SZ\_List$.

**Step 3:**
- **if** $SZ\_List = \emptyset$ **then** STOP
- **if** (max size of search zones in $SZ\_List \leq eps$) **then** STOP
- select one search zone from $SZ\_List$.

**Step 4:**
- update $k = k + 1$
- construct $CEPH^{(k)}$ for the search zone
- update:
  - $PFP^{(k)} = PFP^{(k-1)} \cup \{\text{vertices of } CEPH^{(k)}\}$
  - $I^{(k)} = \bigcup_{(\forall v^i \in PFP^{(k)})} D(v^i)$
  - $E^{(k)} = E^{(k-1)} \setminus [\text{search zone}] \cup CEPH(Y^{(k)})$
- **go to** Step 2

## Conclusion

The proposed algorithm generates the approximated description of the non-convex Pareto frontier combining a number of partial Convex Edgeworth-Pareto Hulls. It can be useful in DSSs based on MOILP with up to 5 criteria.

## References

[1] M. J. Alves , J. Clímaco *A note on a decision support system for multiobjective integer and mixed-integerprogramming problems* European Journal of Operational Research, 155(1), 2004, 258-265.

[2] W. Neely, R. North, J. J. Fortson *An Operational Approach to Multiple Objective Decision Making for Public Water Resources Projects Using Integer Goal Programming.* American Journal of Agricultural Economics, 59(1), 1977, 198-203.

[3] S. Toth, M. McDill *Finding Efficient Harvest Schedules under three Conflicting Objectives.* Forest Science, 55(2), 2009, 117-131.

[4] A. Lotov, V. Bushenkov, G. Kamenev *Interactive Decision Maps. Approximation and Visualization of Pareto Frontier.* Kluwer Academic Publishers, Boston, 2004.