The first boundary value problem for the Laplace equation in unbounded domains

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We consider the first boundary value problem for elliptic systems defined in unbounded domains, which solutions satisfy the condition of finiteness of the Dirichlet integral also called the energy integral

$$\int_{\Omega} |\nabla u|^2 dx < \infty.$$

Basic concepts

Let Ω is an arbitrary open set in $\mathbb{R}^n$. As is usual, by $W^1_{2,\,loc}(\Omega)$ we denote the space of functions which are locally Sobolev, i.e.

$$W^1_{2,\,loc}(\Omega) = \{f : f \in W^1_2(\Omega \cap B_\rho(x)) \\ \forall \rho > 0\,, \forall x \in \mathbb{R}^n\}\,,$$

where B^x_ρ – open ball with center at point x and with radius ρ. We will denote by $\overset{\circ}{W}{}^1_{2,\,loc}(\Omega)$ set of functions from $W^1_{2,\,loc}(\mathbb{R}^n)$, which is the closure of $C^\infty_0(\Omega)$ in the system of seminorms $\|u\|_{W^1_2(\mathcal{K})}$, where $\mathcal{K} \subset \mathbb{R}^n$ are various compacts.

Let $\omega \subseteq \mathbb{R}^n$ is an open set, $\mathcal{K} \subset \omega$ is a compact. We will denote by $\Phi_\varphi(\mathcal{K},\omega)$ the set $\Phi_\varphi(\mathcal{K},\omega) = \{\psi \in C^\infty_0(\omega) : \psi = \varphi$ in the neighborhood of $\mathcal{K}\}$, where equality $\psi = \varphi$ means that $(\psi - \varphi) \in \overset{\circ}{W}{}^1_{2,\,loc}(\mathbb{R}^n \setminus \mathcal{K})$, i.e. $\mu(x)(\psi(x) - \varphi(x)) \in \overset{\circ}{W}{}^1_2(\mathbb{R}^n \setminus \mathcal{K})$, where $\mu \in C^\infty_0(\mathbb{R}^n)$ is an any shearing function.

Let's define a capacitance [3]:

$$\operatorname{cap}_\varphi(\mathcal{K},\omega) = \inf_{\psi \in \Phi_\varphi(\mathcal{K},\omega)} \int_\omega |\nabla\psi|^2 dx\,.$$

The capacitance of arbitrary closed set $E \subset \omega$ in $\mathbb{R}^n$ is defined by the formula $\operatorname{cap}_\varphi(E,\omega) = \sup\limits_{\mathcal{K} \subset E} \operatorname{cap}_\varphi(\mathcal{K},\omega)$. If $\omega = \mathbb{R}^n$, then instead of $\operatorname{cap}_\varphi(E,\mathbb{R}^n)$ we will write $\operatorname{cap}_\varphi(E)$.

Problem statement

The solution of the Dirichlet problem

$$\begin{cases} \triangle u &= 0 \text{ в } \Omega \\ u|_{\partial\Omega} &= \varphi, \end{cases} \tag{1}$$

where $\varphi \in W^1_{2,\,loc}(\mathbb{R}^n)$, is a function $u \in W^1_{2,\,loc}(\Omega)$ such that:

1) $(u - \varphi) \in \overset{\circ}{W}{}^1_{2,\,loc}(\Omega)$, i.e. $\mu(x)(u(x) - \varphi(x)) \in \overset{\circ}{W}{}^1_2(\Omega)$ for any function $\mu \in C^\infty_0(\mathbb{R}^n)$;

2) function $u(x)$ has bounded Dirichlet integral

$$\int_\Omega |\nabla u|^2 dx < \infty\,.$$

Basic results

Theorem 1. *Let's* $\operatorname{cap}_{\varphi - c}(\mathbb{R}^n \setminus \Omega) < \infty$ *for some constant* $c \in \mathbb{R}^n$. *Then the problem (1) has a solution.*

Theorem 2. *Let the problem (1) has a solution and it is true that*

$$\int_{\mathbb{R}^n \setminus \Omega} |\nabla\varphi|^2 dx < \infty\,.$$

Then there is such constant $c \in \mathbb{R}^n$, *that* $\operatorname{cap}_{\varphi - c}(\mathbb{R}^n \setminus \Omega) < \infty$.

References

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