# Optimal control of the welding process

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The new Fast Automatique Differentiation technique of calculating the gradient of the cost function in a controlled system governed by partial differential equations is applied for the optimization of melting and solidification process [1].

The process of melting and solidification of material in a cylindrical domain is described by a two-phase Stefan problem. A heat source with a time-varying strength (the control function) is given. The problem is to find a control function such that no less than a prescribed portion of the sample is melted, solidification proceeds at a rate not exceeding a prescribed magnitude and the total heat supplied by the source is minimal. The heat source is located on the symmetry axis and can be either distributed or point like.

One essential feature of this problem is the moving melting front between two phases. The law of motion of this surface is not known and must be determined. It is the surface where the heat absorption or release associated with the phase transition is taking place. The thermal properties of the phases can be different on both sides of the melting front.

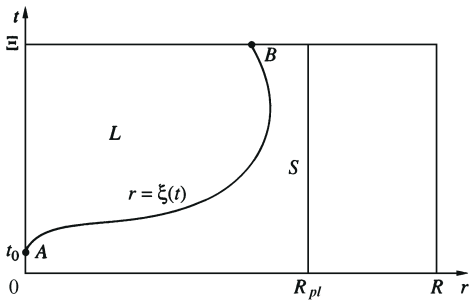


Fig. 1

The radially symmetric time-dependent case of this problem was analyzed. Assuming cylindrical symmetry, then in the plane of the independent variables $(r,t)$ we consider the domain $Q=\{(r,t): 0<r<R, 0<t\le\Theta\}$ (Fig. 1). A smooth curve $AB$ with the equation $r=\xi(t)$ divides $Q$ into two subdomains: $L$ (liquid domain) and $S$ (solid domain). The curve $AB$ is the trajectory of the melting front. Let $t_0\ge 0$ be the time at which $AB$ originates. Then $L$ and $S$ are defined by

$$L=\{(r,t): \quad 0<r<\xi(t), \quad t_0<t\le\Theta\},$$

$$S=\{(r,t): \quad \xi(t)<r<R, \quad 0<t\le\Theta\}.$$

In $Q$ we consider the two-phase Stefan problem

$$\frac{\partial(\rho_L C_L T_L)}{\partial t}-\frac{1}{r}\frac{\partial}{\partial r}\left(rk_L\frac{\partial T_L}{\partial r}\right)-F(r,t)=0, \quad (1)$$

$$(r,t)\in L,$$

$$\frac{\partial(\rho_S C_S T_S)}{\partial t}-\frac{1}{r}\frac{\partial}{\partial r}\left(rk_S\frac{\partial T_S}{\partial r}\right)-F(r,t)=0, \quad (2)$$

$$(r,t)\in S,$$

$$T_S(r,0)=T_{in}(r), \qquad 0<r<R, \quad (3)$$

$$T_L(\xi(t),t)=T_S(\xi(t),t)=T_{pl}, \quad t_0\le t\le\Theta, \quad (4)$$

$$\left[k_S\frac{\partial T_S}{\partial r}\right]\Bigg|_{(\xi(t)+0,t)}-\left[k_L\frac{\partial T_L}{\partial r}\right]\Bigg|_{(\xi(t)-0,t)}=\rho_S\lambda\xi'(t), \quad (5)$$

$$t_0\le t\le\Theta,$$

$$k_S\frac{\partial T_S}{\partial r}\Bigg|_R=\alpha\left[T_{ex}-T_S(R,t)\right], \quad 0<t\le\Theta, \quad (6)$$

$$\frac{\partial T_L}{\partial r}(0,t)=0, \qquad t_0<t\le\Theta, \quad (7)$$

$$\frac{\partial T_S}{\partial r}(0,t) = 0, \qquad 0 < t < t_0. \quad (8)$$

Here, the subscripts $L$ and $S$ denote the liquid and solid phases, respectively;
$T(r,t)$ is the temperature at the point with coordinates $(r,t)$;
$\rho$, $C$, and $k$ are the density, specific heat capacity, and thermal conductivity, respectively;
$\lambda$ is the specific heat of fusion;
$T_{pl}$ is the melting temperature;
$T_{in}(r)$ is the initial temperature of the substance, $T_{in}(r) \le T_{pl}$;
$\alpha$ is the heat exchange coefficient with the surrounding medium;
$T_{ex}$ is the ambient temperature.
The source $F(r,t)$ of input heat can be represented as $F(r,t) = \varphi(r)f(t)$ with $\varphi(r)$ given.

Let $\xi(t)$ be the interface corresponding to the source $f(t)$, $t \in [0,\Theta]$, and let $\xi_f$ be the maximum of $\xi(t)$ over $t_0 \le t \le \Theta$. The function $f(t)$ is said to belong to class $K(\Theta)$ if it satisfies the following conditions:

(I) it is defined and piecewise continuous on $[0,\Theta]$;

(II) it has a piecewise continuous derivative;

(III) it satisfies $0 \le f(t) \le f_{max}$ for all $t \in [0,\Theta]$;

(IV) the corresponding number $\xi_f \ge R_{pl}$, where $R_{pl}$ is given and such that $R_{pl} < R$;

(V) the inequality $\xi'(t) \ge -d^2$ holds for all $t \in [0, \Theta - \beta^2]$.

The variational problem to be solved is posed as follows: among the functions $f(t)$ in $K(\Theta)$, find $f_{opt}(t)$ that minimizes the functional

$$J = \int_0^\Theta f(t)dt. \quad (9)$$

The algorithm that solves the direct problem (determination of temperature distribution and interface separating the phases when control function - supplied heat - is given) is designed to deal with a distributed source, when $\varphi(r) \neq \delta(r)$. Essentially, it is a non front-capturing algorithm.

The main idea of the algorithm was proposed by M. Rose in [2] and was developed by R.E. White in [3], [4]. Here the path of the interface is not regarded as an explicitly imposed interior boundary condition. M. E. Rose suggested a generalized formulation of the problem and shows that classical solution of the problem is its weak solution. On the other hand two classical solutions whose domains of definition are separated by a smooth curve will constitute a weak solution if and only if the Stefan conditions (4), (5) connecting solid and liquid phases on the line takes place.

In accordance with [2] we change from the unknown temperature $T(r,t)$ to the enthalpy function $E(r,t)$ defined in terms of temperature as

$$E(T) = \begin{cases} \rho_S C_S T, & T < T_{pl}, \\ \rho_L C_L (T - T_{pl}) + \rho_S C_S T_{pl} + \rho_S \lambda, & T \ge T_{pl}. \end{cases}$$

Note that the function $E(T)$ has a jump at the melting point $T_{pl}$. Treating the enthalpy $E(r,t)$ as a basic variable and the temperature $T(E)$ as defined by the relation

$$T(E) = \begin{cases} E\rho_S^{-1}C_S^{-1}, & E < E_-, \\ T_{pl}, & E_- \le E \le E_+, \\ \frac{E + (\rho_L C_L - \rho_S C_S)T_{pl} - \rho_S \lambda}{\rho_L C_L}, & E_+ < E, \end{cases}$$

$$(E_- = \rho_S C_S T_{pl}, \quad E_+ = E_- + \rho_S \lambda)$$

one can consider temperature as a continuous function of enthalpy.

In the general case, the heat conductivity depends on temperature and has a jump at the melting point, which corresponds to a transition from solid to liquid phase. In the proposed algorithm, the heat conductivity is a function of enthalpy defined as

$$\Omega(E) = k(T(E)) =$$

$$= \begin{cases} k_S, & E < E_-, \\ k_S + \frac{(E - E_-)(k_L - k_S)}{(E_+ - E_-)}, & E_- \le E \le E_+, \\ k_L, & E > E_+. \end{cases}$$

Problem (1)-(8) is reformulated in terms of the

enthalpy function $E(r,t)$ as

$$\frac{\partial E}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\Omega(E)\frac{\partial T(E)}{\partial r}\right) + F(r,t),\ (r,t) \in Q,$$

$$E(r,0) = E(T_{in}(r)), \qquad 0 < r < R,$$

$$\left.\frac{\partial E}{\partial r}\right|_{r=0} = 0, \qquad 0 \le t \le \Theta,$$

$$\Omega(E)\left.\frac{\partial T(E)}{\partial r}\right|_{r=R} = \alpha\left[T_{ex} - T(E(R,t))\right], 0 \le t \le \Theta.$$

To approximate this boundary value problem in the domain $Q$, we use the implicit approximation with respect to time and the integro-interpolation method ([5]). A nonuniform computation grid was used. The received system of nonlinear algebraic equations was solved by applying iteration and tridiagonal Gaussian elimination ([6]).

The variational problem was solved by reducing the original problem to a nonlinear programming problem. The control function was approximated by a piecewise constant function.

To pick comparison functions from the set of class $K(\Theta)$ functions, we used the method of external penalty functions. In this approach, the set of admissible comparison functions is much broader then $K(\Theta)$, but the cost functional is minimized by an element of the class $K(\Theta)$. This reduces the constraint minimization of the cost functional $J$ in (9) to the unconstraint minimization of the generalized functional $I = J + g(\xi_f) + \Xi$, where $g(r) = A_0(r - R_{pl})^2$ (with a constant $A_0$) is the penalty functional responsible for the fulfillment of the condition $\xi_f = R_{pl}$ and

$$\Xi = \int_0^{\Theta} A(t)\left(\frac{d\xi}{dt} + d^2\right) dt,$$

$$A(t) = \begin{cases} 0, & \left(\frac{d\xi}{dt} + d^2\right) \ge 0, \\ A_0(t), & \left(\frac{d\xi}{dt} + d^2\right) < 0, \end{cases}$$

is the penalty functional ensuring an admissible cooling rate.

The received nonlinear programming problem was solved by various gradient methods. The efficiency of the gradient methods depends essentially on the accuracy which the gradient of the cost functional is calculated with.

An approach to the exact evaluation of cost functional gradients is based on the generalized Fast Automatic Differentiation (FAD) technique [7], which is a natural generalization and development of methods used in nonlinear programming. The generalized FAD technique can be described as follows. The goal of any optimal control problem is to optimize a cost functional depending on controls and state variables. The controls and state variables are connected by certain relations (for example, for given controls, the state variables are determined by solving a boundary value problem for a system of partial differential equations). The first step in the generalized FAD technique involves the discretization of the functional and the constraints. As a result, the cost functional is associated with a function of a finite number of variables, while the constraints are associated with a set of algebraic equations. Thus, we have to optimize a function of several variables that are related by a set of algebraic equations. The second step is to evaluate the gradient of the discrete cost function that is subjected to the constraints.

The FAD technique delivers a unique finite-difference scheme for the adjoint problem and a unique formula for determining the gradient of the cost functional. The value of the gradient of the cost function, calculated according to formulas of the FAD-methodology, is precise for the selected approximation of the optimal control problem. Let us especially note that the machine time needed for calculation the gradient components in the considered problem is not more than half of machine time needed for solving the direct problem.

An analysis of the numerical results obtained suggests the following conclusions about the structure of the optimal control (Fig. 2):

(I) The optimal control consists of two basic components. Figure 2 shows the temporal dependencies of the optimal control obtained by solving the variation problem. The plots correspond to $f_{max} = 10$ and $d^2 = 0.3$.

(II) The first optimal-control component (responsible primarily for melting) coincides with the upper bound $f(t) \equiv f_{max}$.

(III) The second optimal-control component (re-

sponsible for solidification) is smaller than the first (if we compare their averages) and is separated from the latter by a short interval with $f(t) \equiv 0$ .

(IV) The time $t_{on}$ for which the source is turned on at the phase of solidification depends on both $f_{max}$ and the limit cooling rate $d^2$. Depending on these parameters, $t_{on}$ either precedes the time $t_{**}$ at which the extent of the melted domain reaches its maximum possible value $\xi(t_{**}) = R_{pl}$ (for small values of $d^2$), succeeds $t_{**}$ (for large values of $d^2$), or coincides with it.

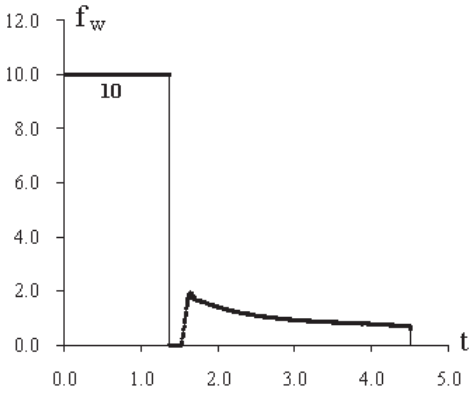


Fig. 2

(V) The second part of the optimal control depends on the parameter $d^2$. This relationship is represented at Fig. 3.

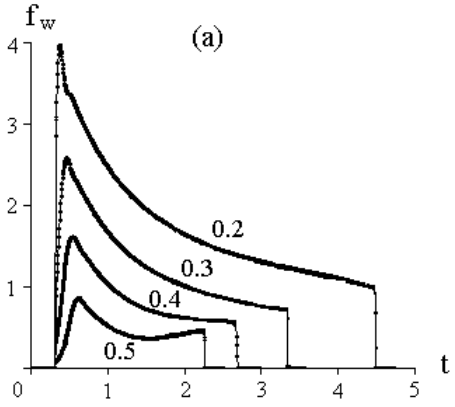


Fig. 3

The investigations of the problem permit to make following conclusions. In the parameter range that was used while investigations took part, the optimal control could be determined from the solution of two successive problems. First, we solve the melting problem and then, using its results as the initial data for the second one, we examine the solidification problem. Usage of such splitting at the solution of the full variational problem essentially economizes expenditures on deriving of the optimal solution.

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## References

[1] A. F. Albou and V. I. Zubov, *Modeling and optimization of melting and solidification process.* BULETINUL ACADEMIEI DE ŞTIINŢE A REPUBLICII MOLDOVA. MATEMATICA, 2004, No. 3(46), p.91–109.

[2] M. E. Rose, *A Method for Calculating Solutions of Parabolic Equations with a Free Boundary* Math. Comput., 1960, No. 14, p.249–256.

[3] R. E. White, *An Enthalpy Formulation of the Stephan Problem.* Math. Comput., 1982, No. 19, p.1129–1157.

[4] R. E. White, *A Numerical Solution of the Enthalpy Formulation of the Stephan Problem.* Math. Comput., 1982, No. 19, p.1158–1172.

[5] A. F. Albu, V. I. Gorbunov and V. I. Zubov, *Optimal control of the process of melting.* Comp. math. math. phys., 2000, No. 40, p.491–504.

[6] A. F. Albu and V. I. Zubov, *A modified scheme for analyzing the process of melting.* Comp. math. math. phys., 2001, No. 41, p.1363–1371.

[7] Y. G. Evtushenko, *Computation of Exact Gradients in Distributed Dynamic Systems.* Optimizat. Methods and Software, 1998, No. 9, p.45–75.