Cyclical growth in a model of decentralized economy

A. P. Abramov*

*Computing Center RAS, apabramov@list.ru

We consider a scheme for a multisectoral decentralized economy to reach a cyclical growth. In our case, the economy is closed and consists of n sectors. The output vector $x(t) \in R^n$ satisfies the inequality $Yx(t) \leq x(t-1)$ at step t, $t = 1, 2, \ldots$, where $Y = \{y_{ij}\}$ is a technological matrix.

As is shown in [1], if the matrix Y is indecomposable and primitive, then the economic system can have a balanced growth $x(t) = \gamma^t x(0)$ at all steps $t \geq 1$ if and only if $x(0)$ is the Frobenius vector of Y. In this case we have $\gamma = 1/\lambda_Y$ where λ_Y is the Frobenius eigenvalue of Y. The balanced growth is also called a turnpike regime.

Further, we consider the case when the vector $x(0)$ is not Frobenius of Y and the system is decentralized. Let us introduce some notation.
By $y_{ji}(t)$ denote the volume of product j that may be used by sector i at step t,
by $x_i^p(t)$ denote the plan of output for sector i at step t (this plan must be fix by the sector before the start of step t) ,
by $x_i^d(t)$ denote the total demand for the product i that was produced at step t,
by $x_i^s(t)$ denote the total sale of the product i produced at step t.

Variables $x_i^d(t)$ and $x_j^p(t+1)$ are related as

$$x_i^d(t) = \sum_{j=1}^{n} y_{ij} x_j^p(t+1), \quad i = 1, \ldots, n.$$

Further we consider a scheme for a decentralized planning.

First, assume that the plan of output of any sector at step t is determined unambiguously by the total sale of products produced at step $t-2$. Specifically, suppose that the economic system adopted the scale, which contains L band for estimation the success of sales. For example, if $L = 5$, this scale might be as follows: level $l = 1$ corresponds to the case when the sales are no more than 60% of output, implementation level $l = 2$ corresponds to the range of (60%70%], ..., level $l = 5$ corresponds to the range (90%100%]. Accordingly, defining the plan to step $t+1$, the sector i uses a formula

$$x_i^p(t) = k_l x_i^s(t-2), \qquad t \geqslant 2, \quad i = 1, \ldots, n, \qquad (1)$$

where coefficient k_l is the same for all sectors at all steps if the sales correspond to the range l. We assume that $k_1 < k_2 < \cdots < k_{L-1} < k_L$.

Second, the produced products are distributed according to the following procedure. Sector i having received orders from consumers calculates coefficient

$$\eta_i(t) = x_i^d(t-1) \,/ x_i(t-1)$$

characterizing endowment of the production plans with the resource that it has produced. Then these coefficients are made known to all sectors of the system. On the basis of the obtained data, each sector (or a certain information center) calculates the maximum value of these indicators:

$$\eta_{\max}(t) = \max_i \eta_i(t).$$

If it turns out that $\eta_{\max}(t) \leq 1$, the demand of each sector for resources is satisfied fully, and volumes of deliveries are enough to fulfil the outlined plans:

$$y_{ij}(t) = y_{ij} x_j^p(t), \quad i, j = 1, \ldots, n.$$

In this case the volumes of production equal the plans: $x_i(t+1) = x_i^p(t+1)$. But if $\eta_{\max}(t) > 1$, then the full fulfilment of plans becomes impossible. In this case all sectors decrease the plan of output using this parameter:

$$x_i^p(t) := x_i^p(t) \,/ \eta_{\max}(t) \,, \quad i = 1, \ldots, n.$$

The demand for all resources decreases accordingly $\eta_{\max}(t)$ times. In this case the corrected plans are fully provided with resources, and the newly calculated indicator $\eta_{\max}(t)$ equals unity.

A plan allowable by resources determines unambiguously the total sale of product i produced at the previous step:

$$x_i^s(t-1) = \sum_{j=1}^{n} y_{ij} x_j^p(t), \quad i = 1, \ldots, n.$$

Then the production cycle begins at step t, after which by (1) the vector $x^p(t+1)$ is determined, etc.

To sum, a necessary condition for functioning of this scheme is the determination of vector $x^p(1)$ and the rule for using the set of parameters $\{k_l\}$.

Theorem 1 *If matrix Y is primitive, vectors $x(0)$ and $x^p(1)$ are strictly positive and inequalities $\sqrt{k_1} \geq \gamma > 1$ hold, then this scheme of planning either brings the system asymptotically to the turnpike, or the normalized sequence of outputs has a finite number of limit points. In the latter case, choosing a neighbourhood of these points is arbitrarily small, we can specify the number of steps, starting from which all members of the sequence will belong to the neighbourhoods repeatedly.*

Denote by m the number of these limit points. If $m > 1$, then it means that starting from step T, we have a cyclical use in a certain order m sets of coefficients $\{k_l\}$. Denote by K_j, $j = 1, \ldots, m$ the diagonal matrix whose non-zero elements are the set of coefficients $\{k_l\}$, appearing at step j of the cycle. In general, a similar matrix, the diagonal elements of which were used in calculating plans for step t denote by $K(t)$, $t \geq 2$. We introduce the parameter $\beta(t)$, which shows what proportion of from initial production plans implemented at step t. It is clear that

$$\beta(t) = \begin{cases} 1, & \eta_{\max}(t-1) \leq 1, \\ 1/\eta_{\max}(t-1), & \eta_{\max}(t-1) > 1. \end{cases}$$

Then at step t we have $x(t) = \beta(t) x^p(t)$. Accordingly, the consumption of products produced at step $t-1$ amount to $x^s(t-1) = Yx(t) = \beta(t) Y x^p(t)$.

Using induction, write the relationship between vectors $x(t)$ and $x^p(1)$:

$$x(t) = \beta(t) \cdots \beta(1) K(t) Y \cdots K(2) Y x^p(1). \quad (2)$$

Let us represent indices for $t \geq T$ as $t = T + ms + j$, where $s = 0, 1, 2, 3, \ldots$, and $j = 0, 1, \ldots, m-1$. Then we can rewrite (2) as

$$x(T+ms+j) = \\ = \begin{cases} \beta(t) \cdots \beta(T+1) M_0^s x(T), & j = 0; \\ \beta(t) \cdots \beta(T+1) K_j Y \cdots K_1 Y M_0^s x(T), & j > 0, \end{cases} \quad (3)$$

where $M_0 \equiv K_m Y K_{(m-1)} Y \cdots K_1 Y$. Using Frobenius eigenvalue λ_0 of matrix M_0, write the latter equality at steps corresponding to the index $j = 0$ as:

$$x(T+ms) = \beta(t) \cdots \beta(T+1) \lambda_0^s \left(\frac{M_0}{\lambda_0} \right)^s x(T). \quad (4)$$

Since the location of zeros in the matrix Y and $K_j Y$, $j = 1, \ldots, m$, coincides then the matrix M_0 is also primitive [2]. Thus [2], the sequence $\left\{ \left(\frac{M_0}{\lambda_0} \right)^s x(T) \right\}$ as $s \to \infty$ converges to the Frobenius vector of matrix M_0. Consequently, the corresponding normalized subsequence $\left\{ \frac{x(\tau)}{\|x(\tau)\|} \right\}$ converges to the normalized Frobenius vector of the matrix M_0, which is denoted by x_0. Because of cyclic presence of multipliers $K_j Y \cdots K_1 Y$ in the second row of (3) it follows that the normalized sequence of vectors $\left\{ \frac{x(t)}{\|x(t)\|} \right\}$ as $t \to \infty$ will cyclically strive to m points on the unit sphere in R^n. These points are uniquely determined by the following vectors:

$$x_0, \quad K_1 Y x_0, \quad \ldots, \quad K_{(m-1)} Y \cdots K_2 Y K_1 Y x_0.$$

If $m = 1$, then the formula (4) takes the form

$$x(T+s) = \beta(t) \cdots \beta(T+1) (\lambda_1)^s \left(\frac{K_1 Y}{\lambda_1} \right)^s x(T), \quad (5)$$

where λ_1 is Frobenius eigenvalue of the matrix $K_1 Y$. In this case we have the normalized sequence of vectors converges to a point on the unit sphere which defined by the Frobenius vector of the matrix $K_1 Y$. In the special case when all diagonal elements of K_1 has the same value k, the equation (5) can be written as follows:

$$x(T+s) = \beta(t) \cdots \beta(T+1) (k \lambda_Y)^s \left(\frac{Y}{\lambda_Y} \right)^s x(T),$$

and the normalized sequence converges to the Frobenius issues vector of the matrix Y. In this case the economic system will asymptomatically reach the turnpike of balanced growth.

References

[1] H. Nikaido, *Convex Structures and Economic Theory.* Academic Press, 1968.

[2] R. A. Horn and Ch. R. Johnson, *Matrix Analysis.* Cambridge University Press, 1986.