Methods for solving the bilevel optimization problems

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1. Introduction. Nowadays, the bilevel optimization problems, arising in various applications [1, 2], seem to be one of the most attractive fields for many experts [1, 3, 4, 5]. Bilevel problems are such optimization problems, which – side by side with ordinary constraints such as equalities and inequalities [6] – include a constraint described as an optimization subproblem, called the lower-level problem or follower problem.

In course of investigation of bilevel optimization problems the difficulty arises already at the stage of defining the concept of solution. The optimistic and pessimistic (guaranteed) definitions of a solution are known to be the most popular [1, 3, 4]. During the three decades of intensive investigation of bilevel optimization problems there were proposed various methods for finding an optimistic solution by different authors (see the survey [5]). But there was a few for finding pessimistic solutions [7, 8].

From the computational point of view difficulties with numerical solving bilevel problems arise due to the fact that even the simpliest linear bilevel problem (with linear goal functions on both levels of hierarchy) in its optimistic setting is an equivalent to a nonconvex optimization problem (in terms of global solution) [4]. So, only results on finding optimistic solutions in bilevel problems of dimension up to 220 (at both upper and lower level variables) are known from literature [9, 10, 11]. As for more difficult problem of finding guaranteed solutions to bilevel optimization problems, only results on numerical solving of illustrative examples of dimension up to 4 are known [8].

In this paper we deal with new methods for finding optimistic and pessimistic solutions to linear and quadratic-linear (where the upper-level goal function is a quadratic function, the lower-level goal function is linear and the feasible sets are polyhydrons) bilevel problems, in which we strive for solving problems of high dimension.

The paper is organized as follows. First, we present a method of solving linear bilevel problems in its optimistic setting based on a reducing of bilevel problem to an optimization problem with d.c. constraint (with nonconvex constraint equality, given by a difference of two convex functions) and Global Search Theory for such problems [12, 13, 14]. Further, we present a method for solving quadratic-linear bilevel problems in its optimistic setting based on a reduction of the initial problem to a family of d.c. minimization problems and Global Search Theory for such problems [12, 13, 15]. Finally, we present an approach for finding guaranteed solutions to quadratic-linear bilevel problems which is based on a reduction of the bilevel problem to a family of auxiliary bilevel problems in optimistic setting and employing the approach for finding optimistic solutions to initial quadratic-linear bilevel problems.

2. Solving the optimistic linear bilevel problem via d.c. constraint problem. Let us consider the following linear bilevel optimization problem in its optimistic setting:

$$\left.\begin{array}{c} \langle c, x\rangle + \langle c_1, y\rangle \downarrow \min\limits_{x,y}, \\ (x,y) \in D \stackrel{\triangle}{=} \{(x,y) \in I\!R^m \times I\!R^n \mid \\ \mid Ax + By \le a,\ \ x \ge 0\}, \\ y \in Y_*(x) \stackrel{\triangle}{=} \operatorname{Argmin}\limits_{z}\{\langle d, z\rangle \mid \ z \in Y(x)\}, \\ Y(x) \stackrel{\triangle}{=} \{y \in I\!R^n \mid A_1 x + B_1 y \le b, \\ y \ge 0\}, \end{array}\right\} (\mathcal{LBP})$$

Replacing the lower-level problem in $(\mathcal{LBP})$ by its necessary and sufficient Karush–Kuhn–Tucker optimility conditions we obtain the problem [1, 4], as follows,

$$\left.\begin{array}{c}\langle c,x\rangle+\langle c_1,y\rangle\downarrow\min\limits_{x,y,v},\\(x,y,v)\in S\stackrel{\triangle}{=}\{(x,y,v)\mid\\\mid Ax+By\le a,\ A_1x+B_1y\le b,\\d+vB_1\ge 0,\ x\ge 0,\ y\ge 0,\ v\ge 0\},\\h(x,y,v):=\langle d,y\rangle-\langle A_1x-b,v\rangle=0.\end{array}\right\}\quad(\mathcal{P})$$

Note that $(\mathcal{P})$ is equivalent to the problem $(\mathcal{LBP})$ in the sense of coincidence of global solutions [4]. It is easy to see that the problem $(\mathcal{P})$ is nonconvex due to the d.c. equality constraint including bilinear function $h(\cdot)$.

Furthermore, for the numerical solving the problem $(\mathcal{P})$ we advanced a global search algorithm [16] based on Global Search Strategy for d.c. constraint optimization problems [12, 14].

Recall that, the global search algorithm consists of two main stages:

1) Local search providing a critical point;

2) Procedures of escaping from critical points, based on Global Optimality Conditions (GOC) [12].

During numerical simulations the global search algorithm showed itself rather effective and promising, for example the all randomly generated linear bilevel problems [18] of dimension up to 1000 were successfully solved [16].

In the next section we describe another approach for finding optimistic solutions to bilevel problems.

3. Solving the optimistic quadratic-linear bilevel problem via d.c. minimization problem. Consider the following quadratic-linear bilevel optimization problem in its optimistic setting:

$$\left.\begin{array}{c}f(x,y)\stackrel{\triangle}{=}\dfrac{1}{2}\langle x,Cx\rangle+\langle c,x\rangle+\\+\dfrac{1}{2}\langle y,C_1y\rangle+\langle c_1,y\rangle\downarrow\min\limits_{x,y},\\(x,y)\in X\stackrel{\triangle}{=}\{(x,y)\in I\!R^m\times I\!R^n\mid\\\mid Ax+By\le a,\ \ x\ge 0\},\\y\in Y_*(x)\stackrel{\triangle}{=}\operatorname*{Argmin}\limits_{z}\{\langle d,z\rangle\mid\ z\in Y(x)\},\\Y(x)\stackrel{\triangle}{=}\{y\in I\!R^n\mid A_1x+B_1y\le b,\\y\ge 0\},\end{array}\right\}(\mathcal{BP})$$

Replacing the lower-level problem with its Karush–Kuhn–Tucker conditions and using penalty method the problem $\mathcal{BP}$ can be reduced to a family of the following problems [1, 17]:

$$\left.\begin{array}{c}\Phi(x,y,v)\stackrel{\triangle}{=}f(x,y)+\mu h(x,y,v)\downarrow\min\limits_{x,y,v},\\(x,y,v)\in S.\end{array}\right\}(\mathcal{P}(\mu))$$

It can be readily seen, that in this case the hidden nonconvexity of the initial problem has been moved to the goal function, and a penalty parameter $\mu>0$ appeared.

Further, for the numerical solving of the problem $(\mathcal{P}(\mu))$ we proposed a global search algorithm [17] based on Global Search Strategy for d.c. minimization problems [12, 13].

Here the main stages of the global search algorithm are ideologically the same as in the previous section.

As to results of numerical simulations with global search algorithm, one can say that all the randomly generated quadratic-linear bilevel problems [18, 19] of dimension up to 300 were solved [17]. So, the approach turned out to be rather successful and promising.

4. Solving the pessimistic quadratic-linear bilevel problem. In this section the following quadratic-linear bilevel optimization problem with guaranteed (pessimistic) solution is considered:

$$\left.\begin{array}{c}\sup\limits_{y}\{F(x,y)\mid y\in Y_*(x)\}\downarrow\min\limits_{x},\\x\in X,\ \ Y_*(x)\stackrel{\triangle}{=}\operatorname*{Argmin}\limits_{y}\{G(y)\mid\\\mid y\in Y(x)\},\end{array}\right\}\quad(\mathcal{BP}_g)$$

where $F(x,y)\stackrel{\triangle}{=}\dfrac{1}{2}\langle x,Cx\rangle+\langle c,x\rangle-\dfrac{1}{2}\langle y,C_1y\rangle+\langle c_1,y\rangle$, $G(y)\stackrel{\triangle}{=}\langle d,y\rangle$, $X\stackrel{\triangle}{=}\{x\in I\!R^m\mid Ax\le a,\ x\ge 0\}$, $Y(x)\stackrel{\triangle}{=}\{y\in I\!R^n\mid A_1x+B_1y\le b,\ y\ge 0\}$.

Along with $(\mathcal{BP}_g)$ let us consider the following auxiliary bilevel optimization problem in its optimistic setting:

$$\left.\begin{array}{c}F(x,y)\downarrow\min\limits_{x,y},\ \ x\in X,\\y\in\operatorname{Argmin}\{G(y)-\nu F(x,y)\mid\\\mid y\in Y(x)\},\end{array}\right\}\quad(\mathcal{BP}_o(\nu))$$

where $\nu > 0$.

Suppose, feasible sets of upper level and lower level are bounded so that

$$(\mathcal{H}) : \quad \begin{array}{c} X \text{ is a bounded set, and} \\ \exists Y : Y \supseteq Y(x) \, \forall x \in X, \, Y \text{ is a compact set.} \end{array}$$

The following assertion develops the corresponding result from [7].

Theorem 1 *[20] Suppose the conditions* $(\mathcal{H})$ *hold, the number sequences* $\{\nu_k\}, \{\tau_k\}$ *tend to zero:* $\nu_k \downarrow 0$, $\tau_k \downarrow 0$. *Let, in addition, the pair* (x^k, y^k) *be an approximate* (τ_k*-)solution to problem* $(\mathcal{BP}_o(\nu_k))$, $k = 1, 2, ...$ *Then any accumulation point of the sequence* $\{x^k, y^k\}$ *turns out to be a pessimistic solution to problem* $(\mathcal{BP}_g)$.

So, in order to solve bilevel problem in pessimistic setting $(\mathcal{BP}_g)$ theorem 1 suggests to solve the series of bilevel problems in optimistic statement $(\mathcal{BP}_o(\nu_k))$ corresponding to the sequence $\{\nu_k\} : \nu_k \downarrow 0$.

Therefore the approaches of the previous sections, were developed to construct the global search algorithm for the problem $(\mathcal{BP}_o(\nu))$ [21], which has been applied for finding a solution to $(\mathcal{BP}_g)$. In the numerical experiments the algorithm was capable to solve all randomly generated test problems of dimension up to 105. Besides for generating the test problems with known solutions we proposed a method [21], developing an approach from [18, 19] in the case of pessimistic solution. Thus, the developed algorithm showed itself rather effective.

5. Conclusion. In the paper we described new methods for finding optimistic and pessimistic solutions to bilevel problems, based on the Global Search Theory. Numerical experiments showed that the methods are capable to solve optimistic and pessimistic bilevel problems of record dimensions. So, the developed approaches and methods turned out to be competitive and operating for bilevel problems.

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