On the optimization of parameters of stabilizers

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This work deals with numerical methods of parameter optimization for asymptotically stable systems. We formulate a special mathematical programming problem that allows us to determine optimal parameters of a stabilizer. This problem involves solutions to a differential equation. We show how to chose the mesh in order to obtain discrete problem guaranteeing the necessary accuracy. The developed methodology is illustrated by an example concerning optimization of parameters for a satellite stabilization system [5, 6, 1]. Some of the results presented here are published in [3].

Consider differential equation

$$\dot{x} = f(x, u),\; x \in R^n, t \geq 0, \tag{1}$$

where $u \in U \subset R^k$ is a parameter. It is assumed that $0 = f(0, u)$ for all $u \in U$ and the zero equilibrium position of system (1) is asymptotically stable whenever $u \in U$. The parameter u should be chosen to optimize, in some sense, the behavior of the trajectories. It is impossible to construct a stabilizer optimal in all aspects. For example, for a linear controllable system, the pole assignment theorem guarantees the existence of a linear feedback yielding a linear differential equation with any given set of eigenvalues. One can choose a stabilizer with a very high damping speed. However, such a stabilizer is practically useless because of so-called pick-effect (see [4, 9, 8]). Namely, there exists a large deviation of the solutions from the equilibrium position at the beginning of the stabilization process, whenever the module of the eigenvalues is big. The aim of this work is to develop a numerical tool oriented to optimization of stabilizer parameters according to different criteria that appear in the engineering practice.

In what follows we denote the set of real numbers by R and the usual n-dimensional space of vectors with components in R by R^n. The space of absolutely continuous functions defined in $[0, T]$ with values in R^n is denoted by $AC([0, T], R^n)$. We denote by $\langle a, b \rangle$ the usual scalar product in R^n and by $|\cdot|$ the Euclidean norm. By B we denote the closed unit ball, i.e., the set of vectors $x \in R^n$ satisfying $|x| \leq 1$. The transpose of a matrix A is denoted by A^*. We use the matrix norm $|A| = \max_{|x|=1} |Ax|$. If P and Q are two subsets in R^n and $\lambda \in R$, we use the following notations: $\lambda P = \{\lambda p \,|\, p \in P\}$, $P+Q = \{p+q \,|\, p \in P, q \in Q\}$.

Statement of the problem. Denote by $x(t, x_0, u)$ the solution to the Cauchy problem

$$\begin{aligned} &\dot{x} = f(x,u),\ x \in R^n,\ t \in [0,T], \\ &x(0) = x_0, \end{aligned} \qquad (2)$$

where u is a parameter from a compact set $U \subset R^k$. Let $f(0,u) = 0$ for all $u \in U$. Consider the functions

$$\varphi_i(u) = \max_{t \in \Delta_i} \max_{x_0 \in B_i} |x(t, x_0, u)|_i,\ i = \overline{0,m}. \qquad (3)$$

Here $\Delta_i \subseteq [0,T]$ are compact sets, $\cdot_i$ are norms in R^n, and $B_i = \{x \in R^n \mid x_i \leq b_i\}$. Consider the following mathematical programming problem

$$\begin{aligned} &\varphi_0(u) \longrightarrow \min, \\ &\varphi_i(u) \leq \bar{\varphi}_i,\ i = \overline{1,m}, \\ &u \in U. \end{aligned} \qquad (4)$$

Many problems of stabilization systems' parameters optimization can be written in this form.

Minimization of the final deviation. The problem is to determine the optimal values of the system parameters that guarantee minimal deviation of the system state from the zero equilibrium position at the final moment of time. This problem can be formalized as follows:

$$\begin{aligned} &\max_{x_0 \in B} |x(T, x_0, u)| \longrightarrow \min, \\ &u \in U \end{aligned}$$

For linear systems $\dot{x} = A(u)x$ with $T \gg 1$, this problem is an approximation for the maximization of the degree of stability (see [10]).

Minimization of the maximal deviation. This problem consists in determination of parameters that correspond to minimization of the maximum deviation of trajectories and satisfy certain restrictions at the final moment of time. This problem can be formalized as follows:

$$\begin{aligned} &\max_{t \in [0,T]} \max_{x_0 = 1} x(t, x_0, u) \to \min, \\ &\max_{x_0 = 1} x(T, x_0, u) \leq \delta, \\ &u \in U. \end{aligned}$$

Both of the above problems are of interest for stabilization theory and have form (4). Problem (4) has some special features and its study can be useful for stabilization systems parameters optimization; however its analytical study can hardly be performed for more or less complex systems. For this reason, we focus on the numerical aspects of this problem.

Numerical methods. Let $\varepsilon > 0$ be small enough. We approximate problem (4) by the following problem

$$\begin{aligned} &\bar{\varphi}_0 \longrightarrow \min, \\ &\tilde{x}(t_k^i, x_j^i, u)_i \leq \bar{\varphi}_i + \varepsilon,\ i = \overline{0,m}, \\ &u \in U, \end{aligned} \qquad (5)$$

where $t_0^i = 0$, $t_k^i \in \Delta_i$, $x_j^i \in B_i$, $j = \overline{1,J}$, and $\tilde{x}(t_{k+1}^i, x_j^i, u) = \tilde{x}(t_k^i, x_j^i, u) + \tau f(\tilde{x}(t_k^i, x_j^i, u), u)$, $\tau = t_{k+1}^i - t_k^i$, $k = \overline{0,N}$, is the Euler approximation for the solution $x(\cdot, x_j^i, u)$. Problem (4) can be approximated by problems (5) with any given accuracy.

Assume that

$$f(x,u) = A(u)x + g(x,u),$$

where matrix $A(u) = \nabla_x f(0,u)$ has eigenvalues with negative real part and the function $g(\cdot, u)$ satisfies $g(0,u) = 0$ and the Lipschitz condition $\mathrm{g}(\mathrm{x}_1, u) - g(x_2, u) \leq L_g^u \max\{|x_1|, |x_2|\}\, x_1 - x_2$ with $L_g^u > 0$ for all x_1 and x_2 in a neighborhood of the zero equilibrium position. Consider functions $\varphi_i(\cdot)$ defined by (3), assuming that the balls B_i are contained in a sufficiently small neighborhood of the origin. Consider $\delta > 0$. Let $K_i(\delta)$ and $J_i(\delta)$ be sets of indices such that the points $t_k^i \in \Delta_i$, $k \in K_i(\delta)$, and $x_j^i \in B_i$, $j \in J_i(\delta)$ form a δ-net in Δ_i and B_i, $i = \overline{1,m}$, respectively. Define the functions

$$\varphi_i^\delta(u) = \max_{k \in K_i(\delta)} \max_{j \in J_i(\delta)} \tilde{x}(t_k^i, x_j^i, u),\ i = \overline{0,m}.$$

Problem (5) can be written as

$$\begin{aligned} &\varphi_0^\delta(u) \longrightarrow \min, \\ &\varphi_i^\delta(u) \leq \bar{\varphi}_i + \varepsilon,\ i = \overline{1,m}, \\ &u \in U. \end{aligned} \qquad (6)$$

Denote by $\hat{u}$ and u^δ the optimal parameters for problems (4) and (6), respectively.

For any $\epsilon > 0$ there exists $\delta > 0$ such that u^δ is an admissible solution to the following problem

$$\begin{aligned}&\varphi_0(u) \longrightarrow \min,\\&\varphi_i(u) \leq \bar{\varphi}_i + 2\varepsilon,\ i = \overline{1,m},\\&u \in U,\end{aligned}$$

and

$$\varphi_0(u^\delta) \leq \varphi_0(\hat{u}) + 2\varepsilon.$$

This theorem allows one to choose the parameters of discretization in order to obtain optimal stabilizer parameters with a necessary precision. A rigorous formulation of this claim is the following. Denote by $V(\sigma)$ the value of the problem

$$\begin{aligned}&\varphi_0(u) \longrightarrow \min,\\&\varphi_i(u) \leq \bar{\varphi}_i + \sigma,\ i = \overline{1,m},\\&u \in U.\end{aligned}$$

Assume that problem (4) is *calm* in Clarke's sense (see [2]). Then, there exists a constant $K > 0$ satisfying the inequality

$$\frac{V(2\varepsilon) - V(0)}{2\varepsilon} > -K,$$

for all $\varepsilon > 0$ sufficiently small. It follows from Theorem that

$$\tilde{V} - V(0) \leq M\varepsilon,$$

where $\tilde{V} = \varphi_0(\tilde{u})$, $\tilde{u}$ is the solution of problem (5), and $M = 2\max\{1, K\}$.

The exact formulas for $\delta = \delta(\epsilon)$ leading to the proof of Theorem are presented in Appendix. The main tool used to obtain them is the following version of Filippov-Gronwall inequality [7].

Let $P = \{p \in R^n |\ \langle p, Vp\rangle \leq 1\}$, where V is a symmetric positive definite matrix. Consider the functions $y(\cdot) \in AC([0,T], R^n)$ and $\xi(\cdot) \in AC([0,T], R)$, $\xi(t) \geq 0$, satisfying the following condition

$$\max_{\langle p, Vp\rangle = 1} (\langle \dot{y}(t), Vp\rangle - \langle f(y(t) - \xi(t)p), Vp\rangle) \leq \dot{\xi}(t),$$

for almost all $t \in [0,T]$. Then $x(t) \in y(t) + \xi(t)P$ for all $t \in [0,T]$, whenever $x_0 \in y(0) + \xi(0)P$, where $x(t)$ is the solution to the Cauchy problem $\dot{x} = f(x)$, $x(0) = x_0$.

Note that the use of this theorem allows us to obtain a more precise estimates for the number of points in the meshes needed to achieve a given discretization accuracy. The estimates based on the usual Gronwall inequality can be significantly improved for asymptotically stable systems if we take into account the behaviour of the trajectories for large values of time. Theorem is a natural tool for this. For example, according to the classical estimates, the number of points in the mesh in t needed to esure a given precision, grows exponentially with the length of the time interval. However, the estimates obtained from Theorem for asymptotically stable systems give a linear growth of the number of points in the mesh. This is of practical importance. Optimization problem (6) is a hard nonsmooth problem. Our computational experience shows that the NelderMead method is the most adequate method to solve it. The numerical solution of this problem significantly depends on the structure of the involved functions. Sometimes the computational effort is very serious. However, the problem of optimal choice of parameters is solved only once, at the stage of the control system development, and the time needed for its solution is not so important. Moreover, our estimates for the number of points of discretization allow us to construct an adequate mesh and to significantly reduce the CPU time.

The methods usually applied to optimize the parameters of a stabilization system are based on the idea of the maximum stability degree, that is, the minimization of the system's transition time. These methods, however, face the problem of so-called peak effect when the deviation of the system trajectory from the equilibrium increases with the decrease of the time of response. The approach suggested in this work (see also [3]) consists in a numerical analysis of a stabilization system based on a more complete and flexible description of the system behaviour capable to take into account not only the stability degree, but also the maximum deviation of the trajectory on a given time interval or at a given moment. For this optimization problem, we develop a numerical method and prove that it can guarantee a given accuracy for the problem solution. This method is applied to optimization of a stabilization system for a satellite with a gravitational stabilizer [5, 6, 1]. The obtained results show that the above approach can offer solutions more adequate for practical implementation than those given by optimization of the stability degree.

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