# On integer points in polyhedra *

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1. Let $a_{ij}$ be integer

$$(i = 0, \cdots, m; j = 0, \cdots, n),$$

$$\alpha = max\{|a_{ij}|,\ i = 1, \cdots, m;\ j = 1, \cdots, n\},$$
$$\beta = max\{|a_{ij}|,\ i = 0, \cdots, m;\ j = 1, \cdots, n\},$$
$$\gamma = max\{|a_{ij}|,\ i = 1, \cdots, m;\ j = 0, \cdots, n\}.$$

Suppose that $a_{i.} = (a_{i1}, \cdots, a_{in})$ is the $i$-th row $(i = 1, \cdots, m)$ of the matrix $A; a_{.1}, \cdots, a_{.n}$ are its columns; $r$ is rank of the matrix $A; A(I, I) = (a_{ij})$ is submatrix of the matrix $A$, where $i \in I \subseteq \{1, \cdots, m\}$, and $j \in J \subseteq \{1, \cdots, n\}$.

We consider the system in the rational variables $x_j$ $(j = 1, \cdots, n)$

$$\sum_{j=1}^{n} a_{ij}x_j \ \leq\ a_{i0} \quad (i = 1, \cdots, m), \tag{1}$$

the set $P$ of its solutions and the set $M = Z^n \bigcap P$ of integer points of the polyhedron $P$.

Let $C$ be set of the rational solutions of the homogeneous system

$$u_0 a_{0j} - \sum_{j=1}^{m} u_i a_{ij} = 0 \ (i = 1, \cdots, n) \tag{2}$$

$$u_i \geq 0 \ (i = 0, \cdots, m). \tag{3}$$

It is well known [1,2] that extremal vectors of cone $C$ are connected with optimal plans of the couple of duality linear programming problems (LPP) and there exists a polynomial algorithm for following problem

Pr 0: to get $x \in P$ or to prove that $P = \oslash$.

Let us consider the problem

Pr 1: to get $x \in M$ or to prove that $M = \oslash$.

No algorithms are known for solution of $P_1$ which is bounded from above by a polynomial of the lengt $l$ of the input date where $l$ may be defined as $mn \log(\beta + 2)$.

**Theorem 1** [1, 2]. *The problem Pr1 is NP-hard.*

2. Now we shall assume that $r = n$. Then convex hull $convM$ of the set $M$ is polyhedron too, so, it can be described by a finite system of linear inequalities and its vertex set $N$ is bounded. Let $F$ be facet set of the $convM$.

**Theorem 2** [2]. *If $m = n$, then there is a polynomial algorithm for finding $x \in N$.*

**Theorem 3** [3]. *If $m = n + 1$, then the problem $P_1$ is polynomially complete.*

Therefore, it's reasonable to call this problem to be simplest complete problem among integer linear programming problems.

In connection with the Theorem 1 the paper [4] introduced the concept of a *pseudopolynomial* algorithm, that requires a time bounded by a polynomial in $n$ and $\beta$ (but not in $\log \beta$).

For any fixed $m$ it is not hard to construct [5] a pseudopolynomial algopithm $\mathcal{A}_1$ for solving the problem $P_1$ by using the method of dynamic programming.

We call an algorithm quasipolynomial if for any $n$ there exists a polynomial $f_n(x)$ such that when being applied to any $n$-dimensional problem which absolute values do not exceed $\alpha$ the algorithm has complexity that is bounded above by $f_n(log\alpha)$. In 1981 Lenstra [6] proposed a quasipolynomial algorithm $\mathcal{A}_2$ for the problem $Pr1$.

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**Theorem 4** [3]. *If $r = n$, then for any fixed $n$*

*1) the number $|N|$ is bounded above by a polynomial of degree $n$ in $\sqrt{m}$ and $\log \alpha$,*

*2) the numbers $|N|$ and $|F|$ are bounded above by some polynomial in $n$ and $\log \alpha$.*

We consider the problem

$Pr2$: to get $N$,

and the problem

$Pr3$: to get $F$.

**Corollary.** There are a quasipolynomial algorithm for $Pr2$ and $Pr3$.

For a natural number $d$ we denote by $A(d)$ the set of matrices $A \in \mathbf{Z}^{m \times n}$ for which any minors does not exceed $d$ in absolute value.

The new results are the following theorems.

**Theorem 5** *If a matrix $A$ is in $A(d)$ and rank of $A$ is equal to $n$, then there exists a polynomial algorithm for the problem $Pr1$.*

**Theorem 6** *If a matrix $A$ is in $A(d)$ , rank of $A$ is equal to $n$ and $m = n + 1$, then there exists a polynomial algorithm for the problem $Pr3$.*

This results are formulated as the hypothesis in [3].

In prooving the Theorem 5 we used [7 - 13] and the following result.

**Lemma 1** [3]. *The change problem is polynomially complete.*

## References


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