The estimation of the yield of the pool of investment projects in the optimal investing problem for continuous time

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How to increase the investment attractiveness is the one of the most important question for Russian deeconomy today. Not only depressed manufacturing sector but also developed energy sectors need large investments for technical upgrade and development. For example oil sector needs investments for oil output support and utility sector needs investments for generating capacity renovation. The key factor for investments decisions is the yield that investor could earn. This article considers the problem of the estimation of the yield of pool of renewable investment projects. The Cantor–Lippman approach [1] was used for solving such a problem. One of the main objective of our work was to consider the general case when the cash flows from investments are irregular. So we introduce the formulation of the Cantor–Lippman model for continuous time.

We assume that investor could realize his opportunities by undertaking investment projects. These projects are stationary and could be realized at any scale. Investor has a pool of projects with M projects in it. Each project is completely described by its cash flow. The cash flow of the project k is described by the function $\Phi^k(t)$ valuing the cash flow balance by the time t. Each project has the finite time of realization T^k. So we could define the common time of projects realization as $T = \max\limits_k T^k$.

These assumptions could be formalized in the following form. The investment project cash flow is described by the signed measure χ^k that is defined on Borel σ-algebra $\mathfrak{B}$ on the reals. The signed measure χ^k is supported in a such set S^k that $Conv(S^k) \subseteq [0;T]$. We will use Jordan decomposition $\chi^k = \chi_1^k + \chi_2^k$, and Hahn decomposition $[0;T] = A_k^- \cup A_k^+$. The sets $A_k^- \in \mathfrak{B}$ and $A_k^+ = [0;T] \setminus A^-$ are positive set and negative set accordingly for the signed measure χ^k, that is $\forall B \in \mathfrak{B}\ \chi^k(A_k^- \cap B) \leq 0$, $\chi^k(A_k^+ \cap B) \geq 0$, and $\chi_1^k = \chi^k([0;T] \cap A_k^+)$, $\chi_2^k = \chi^k([0;T] \cap A_k^-)$. This decomposition have an intuitive interpretation. The measure χ_1^k defines costs balance for the project and χ_2^k defines return balance for the project. We will use distribution function $\Phi^k(t) = \chi^k((-\infty;t))$. Under our assumption the function $\Phi_1^k(t)$ is a monotonically nonincreasing function and the function $\Phi_2^k(t)$ is a monotonically nondecreasing function and $\Phi_i^k(x) = 0$ for $x \leq 0$, $\Phi_i^k(x) = \Phi_i^k(T)$ for $x > T$, $i = 1, 2$.

We can assume, without limiting the generality, that any investment project starts with a homogeneous cash flow: $\forall k \in [1;M]\ \exists \tau_k > 0 : \forall t \in [0;\tau_k]$ or $\Phi_1(t) < \Phi_1(0)$, or $\Phi_2(t) > \Phi_2(0)$.

Any investment project in the pool has constant return to scale. Investor could manage projects by choosing the intensity of their realization. Let's denote the intensity of realization of the project k at time t as $u^k(t)$. Assume that $u^k(t) \in L_\infty$. So the cash flow to investor at time $(t+\tau)$ equals $u^k(t) \cdot d\Phi^k(\tau)$.

The investor's state is described by $s(t)$ his account balance at time t . Assume that $s(0) > 0$. Investor's account balance changes due to realization of all projects that was started in $[0;t]$ time frame, so

$$s(t) = s(0) + \sum_{k=1}^{N} \int_0^t \Phi^k(t-x)u^k(x)dx.$$

Assume that investor plans to fix earnings at the time $\hat{T}$. Thus investor stops to invest from the time $\hat{T}-T$, i.e. $u^k(t) = 0 \ \forall\, k = 1, \dots, N$ for $t \geq \hat{T} - T$.

The investment opportunities including loans opportunities are completely described by the pool of investment projects. Investor must maintain nonnegative account balance.

Under our assumptions we could describe investor's optimal investing problem by the following system:

$$\begin{cases} s(t) = s(0) + \sum\limits_{k=1}^{N} \int\limits_0^t u^k(x) \cdot \Phi^k(t-x)dx, \\ 0 \leq t \leq \hat{T}, \\ s(t) \geq 0,\ u^k(t) \geq 0,\ u^k(t) \in L_\infty, \\ \forall t \geq 0,\ \forall k = 1 \dots N, \\ u^k(t) = 0,\ t \geq \hat{T} - T,\ \forall k = 1 \dots N, \\ s(\hat{T}) \to \sup\,. \end{cases} \tag{1}$$

The system (1) is the analog of Cantor–Lippman model for continuous time. Following Cantor–Lippman approach we could define the yield of the pool of investment projects as the capital growth rate under (1). It means that if $V(\hat{T})$ is the optimal value of the functional in the system (1) with the time horizon $\hat{T}$, then the yield could be defined as $\lim\limits_{\hat{T}\to\infty} \dfrac{\ln V(\hat{T})}{\hat{T}}$.

Let's define $\tilde{\phi}^k(p) = \int\limits_0^T e^{-pt} d\Phi^k(t)$, $F(p) = \max\limits_k \left\{ \tilde{\phi}^k(p) \right\}$. The function $\tilde{\phi}^k(p)$ is the net present value (NPV) for investment project with the cash flow balance function Φ^k and the discount rate p.

Definition 1. *The pool of investment projects has an arbitration cash flow structure if $\forall\ p \geq 0$ $F(p) > 0$.*

Definition 2. *The pool of investment projects has an ineffective cash flow structure if $F(0) \leq 0$.*

Definition 3. *The pool of investment projects has a standard cash flow structure if it has neither arbitration nor ineffective cash flow structure.*

Theorem. *In the problem (1)*

1. *If there is a pool of investment projects with a standard cash flow structure then $\exists p^* = \min\{p > 0 \,|\, F(p) = 0\}$ and $\lim\limits_{\hat{T}\to\infty} \dfrac{\ln V(\hat{T})}{\hat{T}} = p^*$.*
2. *If there is a pool of investment projects with an ineffective cash flow structure then $\lim\limits_{\hat{T}\to\infty} \dfrac{\ln V(\hat{T})}{\hat{T}} = 0$.*
3. *If there is a pool of investment projects with an arbitration cash flow structure then $\exists \hat{T}_0 : \forall \hat{T} > \hat{T}_0 : V(\hat{T}) = +\infty$.*

According to the Theorem we can define $\lim\limits_{\hat{T}\to\infty} \dfrac{\ln V(\hat{T})}{\hat{T}} = +\infty$ for a pool of investment projects with an arbitration cash flow structure.

The Theorem states that the cash-flow based classification reflects difference in the yield. The cash-flow based classification is defined by the sign of the function $F(p)$. If the function $F(p)$ is positive for all $p \geq 0$, then the pool can realize an arbitrage strategy, i.e. generate income without costs. Such a pool can generate any capital growth rate for investor. If the function $F(p)$ is nonpositive at zero then the simple (non discounted) sum of cash flow for each project in the pool is nonpositive. It means that all projects in the pool are ineffective and the yield equals zero. The most interesting case is the pool with standard cash flow. This case can be described by the two conditions. The first condition is that the function $F(p)$ is positive at zero. It means that there is at least one effective project in the pool. The second condition is that the function $F(p)$ has at least one positive root. It means that the pool does not realize arbitrage. The minimal positive root of the function $F(p)$ is the yield for pool with standard cash flow.

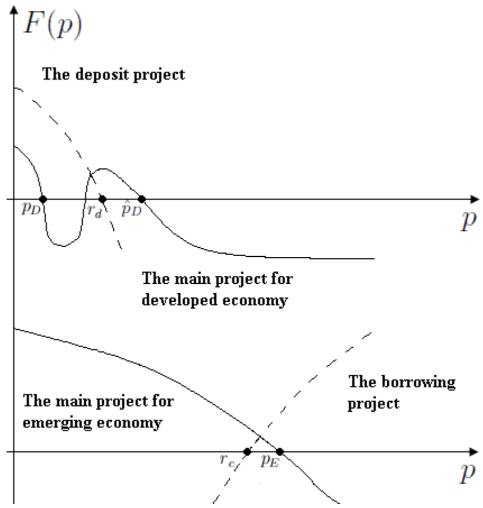

Figure 1.

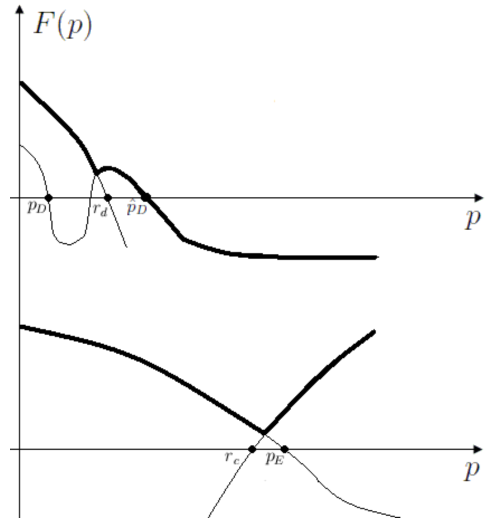

Figure 2.

We consider the one example of the result implementation where the cooperation of a developed and an emerging economy is analyzed. Each economy is characterized by the main investment project describing production in a real sector. The main project for developed economy has the internal yield p_D (see the full line on the upper part of the figure 1). The main project for emerging has internal yield p_E (see the full line on the lower part of the figure 1). Since the production costs for developed economy are higher we assume that $p_E > p_D$. The cooperation between economies are realized by deposit and borrowing projects. The deposit project at the interest rate r_d is available for developed economy and reflects the capital outflow to emerging economy (see the dashed line on the upper part of the figure 1). This project enhance the yield for developed economy to $\hat{p}_D$. You can see it at the upper part of the figure 2 where the bold line is the upper envelope for the main and the deposit project in developed economy.

The capital from developed economy is transferred by a financial intermediary to the emerging economy in a form of loan. So the borrowing project at the interest rate r_c is available to the emerging economy (see the dashed line on the lower part of the figure 1). The interest rate r_c is higher than r_d reflecting risk associated with investing in the emerging economy. The financial intermediary tries to increase the rate r_c in order to maximize his margin $(r_c - r_d)$. At the same time the rate r_c can not be more than or equal p_E because otherwise the total yield (the first intersection of the upper envelope line and the p-axis) for the emerging economy will not change and the main project will be the only one which implemented. So the rate r_c is higher than p_E. You can see this situation at the lower part of the figure 2 where the bold line is the upper envelope for the main and the borrowing project in the emerging economy. As the upper envelope lies entirely in the positive half this situation is an arbitrage. But this situation is unstable. Unconstrained arbitrage realization means borrowing aggressively. As the debt of emerging economy increase the cost of debt will increase, i.e. the rate r_c will increase. The debt will grow until the rate r_c will increase to p_E. Then the borrowing process will stop and the economy switch to the main project only. The constrained arbitrage means that emerging economy develops through limited borrowing from developed economy. Developing emerging economy followed by increasing production costs and increasing credit quality of the emerging economy. Increase in production costs means decreasing the internal yield p_E. Increasing credit quality means decreasing the rate r_c. So the development process can be stable. The development process goes until the emerging economy will become developed.

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References

[1] Cantor D.G., Lipman S.A. Investment selection with imperfect capital markets // Econometrica, 1983, Vol. 51, N 4, p.1121–1144.

[2] Cantor D.G., Lipman S.A. Optimal Investment Selection with a Multitude of Projects// Econometrica, 1995, Vol. 63, N 5, p.1231–1240.