On one branching approach in branch-and-bound method for solving integer linear programming problems

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In the paper the integer linear programming problem (ILP) in standard form is considered:

$$\max\ (c, x)$$

$$\begin{cases} Ax \leqslant b \\ x \in \mathbb{Z}_+^n \end{cases}$$

$$A \in \mathbb{Z}^{m\times n},\ b \in \mathbb{Z}^m,\ c \in \mathbb{Z}^n,$$

where $\mathbb{Z}$, $\mathbb{Z}_+$, $\mathbb{Z}^m$, $\mathbb{Z}^{m\times n}$ are set of integers, set of non-negative integers, integer columns of size m, integer $m \times n$ matrices correspondingly.

One of the common methods for solving ILP is well-known branch-and-bound algorithm [1, 4] that solves initial problem by dividing it to smaller subproblems. The algorithm uses finiteness of feasible solutions set and it performs directed search by evaluation and rejection unpromising variants.

The main idea of the suggested method can be formulated as follows. Let P' be LP-relaxation of current subproblem and $\tilde{x}$ be optimal solution of P'. In classical branch-and-bound method [2] non-integer component $\tilde{x}_i$ is selected and subproblems P'_0 and P'_1 are generated by adding inequalities $x_i \leqslant \lfloor \tilde{x}_i \rfloor$ and $x_i \geqslant \lceil \tilde{x}_i \rceil$. We suggest to produce P'_0 and P'_1 subproblems by adding common inequalities $\sum\limits_{i=1}^{n} a_{1i}x_i \leqslant b_1$ and $\sum\limits_{i=1}^{n} a_{2i}x_i \geqslant b_2$ such that

$$\sum_{i=1}^{n} a_{1i}\tilde{x}_i > b_1, \quad \sum_{i=1}^{n} a_{2i}\tilde{x}_i < b_2 \tag{1}$$

$$\forall x\colon Ax \leqslant b, x \in \mathbb{Z}_+^n \Rightarrow$$
$$\left(\sum_{i=1}^{n} a_{1i}x_i \leqslant b_1\right) \vee \left(\sum_{i=1}^{n} a_{2i}x_i \geqslant b_2\right) \tag{2}$$

Conditions (1) denote that optimal non-integer vector $\tilde{x}$ isn't in feasible region of P'_0 and P'_1 subproblems, conditions (2) guarantee the absence of feasible integer points in region constrained by inequalities $\sum\limits_{i=1}^{n} a_{1i}x_i > b_1$ $\sum\limits_{i=1}^{n} a_{2i}x_i < b_2$.

The method of generating common inequalities that satisfy to (1), (2) is based on cutting theory and theory of linear comparison systems [3].

Let $B \in \mathbb{Z}^{n\times n}$, $b \in \mathbb{Z}^n$, and $W = \{x \in \mathbb{R}^n \mid Bx \leqslant b\}$. We assume that matrix B is nonsingular, $\Delta = |B| \neq 0$. In this case there is the only extreme point $w = B^{-1}b$ in the set W.

Denote $\mathcal{B} = \{u\colon uB \equiv 0 \pmod{\Delta}\}$, $\bar{\mathcal{B}} = \{u \in \mathcal{B}\colon 0 \leqslant u_i < \Delta,\ i = 1, \ldots, n\}$. One of the common approaches to describe $\bar{\mathcal{B}}$ is to use Smith normal diagonal form of the matrix B. Let D be Smith normal form for matrix B such that $D = PBQ$, where P, Q are unimodular matrices and $Q = (q_1, \ldots q_n)$, $D = \operatorname{diag}(d_1, \ldots d_n)$, then

$$\bar{\mathcal{B}} = \left\{ u = \operatorname{res}_\Delta \left(\sum_{i=1}^{n} \frac{\Delta}{d_i} \beta_i q_i \right) : 0 \leqslant \beta_i < d_i \right\}$$

It is known [3] that if $u \in \mathcal{B}$, $u > 0$, $ub \not\equiv 0 \pmod{\Delta}$ then inequality $\frac{uBx}{\Delta} \leqslant \left\lfloor \frac{ub}{\Delta} \right\rfloor$ is regular cut that cuts extreme point w from set W. If we omit limitation on non-negativeness of vector u components then we get two inequalities:

$$\frac{uBx}{\Delta} \leqslant \left\lfloor \frac{ub}{\Delta} \right\rfloor \tag{3}$$

$$\frac{uBx}{\Delta} \geqslant \left\lceil \frac{ub}{\Delta} \right\rceil \tag{4}$$

which satisfy to (1), (2).

So branching algorithm can be formulated as follows. Let $\tilde{x}$ is solution of current subproblem P':

$$\max\ (c, x)$$
$$A'x \leqslant b'$$

where matrix A' contains both constraints of the initial problem (including non-negativeness of the vector x components) and new branching inequalities.

Branching includes next steps:

1. Select submatrix $B \in \mathbb{Z}^{n\times n}$ and vector $b \in \mathbb{Z}^n$ such that $\tilde{x} = B^{-1}b$. It is possible as $\tilde{x}$ is extreme point for feasible region of the problem.

2. Calculate Smith normal form for matrix B: $D = PBQ$.

3. Select vector $u \in \mathcal{B}$ using (ADD REF) $\mathcal{B}$: coefficients $\beta_1, \ldots \beta_n$ are selected from sets $\{0, \ldots, d_{11} - 1\}, \ldots, \{0, \ldots, d_{nn} - 1\}$ accordingly.

4. Select subset of variables $N \subset \{1, \ldots, n\}$ with cardinality k: $1 \leqslant k \leqslant n - 1$. $\forall i \in N$: $u_i := u_i - \Delta$.

5. Create 2 new subproblems by adding inequalities $\frac{uBx}{\Delta} \leqslant \left\lfloor \frac{ub}{\Delta} \right\rfloor$ and $\frac{uBx}{\Delta} \geqslant \left\lceil \frac{ub}{\Delta} \right\rceil$ to constraints of P' subproblem.

The question how to select vector u on step 3 and subset N is still open.

Experimental evaluation of the suggested method has been done. There are several examples of ILP problems where the algorithm performs less iterations versus classical branch-and-bound approach.

References

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