About Methods for Solving Quasi Variational Inequalties

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Let we denote by H a Hilbert space. By $\pi_C(x)$ we denote the Euclidian projection of point x onto the set C.

Let $C : H \to 2^H$ be a set-valued mapping with nonempty closed and convex values. Consider a continuous operator $F(x) : C \to H$, which is strongly monotone

$$\langle F(x)-F(y), x-y\rangle \geq \mu\|x-y\|^2, \ \ \forall\, x, y \in C, \quad (1)$$

and Lipschitz continuous on C

$$\|F(x) - F(y)\| \leq L\|x - y\|, \ \ \forall\, x, y \in C. \quad (2)$$

If $\mu = 0$, then F is a monotone operator, so we always assume $\mu > 0$.

The problem of our interest is the following quasi-variational inequality (QVI):
find $x_* \in C(x_*)$ such that

$$\langle F(x_*), y - x_*\rangle \geq 0, \ \forall y \in C(x_*). \quad (3)$$

For solving problem (3), one can use continuous gradient-type method:

$$x'(t) + x(t) = \pi_{C(x(t))}[x(t) - \lambda(t)F(x(t))],$$

for $t > 0$, $x(0) = x_0$, where x_0 is a given initial point. We will prove convergence of this method and convergance rate when the sets $C(x), x \in H$ satisfies:
exists $\alpha < \frac{\mu^2}{L(L+\sqrt{L^2-\mu^2})}$ such that

$$\|\pi_{C(x)}(z) - \pi_{C(y)}(z)\| \leq \alpha\|x - y\|, \ \ \forall\, x, y, z \in H.$$

We also discuss the case when the sets $C(x), x \in H$ are given by

$$C(x) := c(x) + \overline{C}, \quad (4)$$

where $\overline{C}$ is a closed convex set and $c : H \mapsto H$ is a Lipschitz continuous function with Lipschitz constant l. In this case, the corresponding continuous gradient-type method has the form

$$x'(t) + x(t) =$$

$$c(x(t)) + \pi_{\bar{C}}[x(t) - c(x(t)) - \lambda(t)F(x(t))],$$

for $t \geq 0$, $x(0) = x_0$. Then the following theorem has been proved:

Theorem 1 *If operator F satisfied (1) and (2), multifunction $C : H \mapsto 2^H$ is given by (4) and conditions $2 - \alpha(t)(L + \mu) > 0$ and $A(t) = -l^2 - 2lL\alpha(t) - \mu^2\alpha^2(t) + 2\mu\alpha(t) \geq A_0 > 0$, are satisfied, then the continuous gradient-type method converges to the unique solution of problem (3) with the following rate*

$$\|x(t) - x_*\|^2 \leq \exp\left(-\int_{t_0}^{t} A(t)dt\right)\|x_0 - x_*\|^2.$$

In this paper, we also describe a continuous second-order method for solving QVI:

$$\beta(t)x''(t)+x'(t)+x(t) = \pi_{C(x(t))}(x(t)-\lambda(t)F(x(t))),$$

$x(0) = x_0, \quad x'(0) = x_1$, where parameters of method $\lambda(t)$ and $\beta(t)$ are continuous, nonegative functions for all $t \geq t_0$ and x_0, x_1 are initial points from H.

If the sets $C(x)$ are given as in (4) we have proved the following theorem:

Theorem 2 *Suppose that the following conditions are fullfied:*

1) *Operator F satisfied (1) and (2);*

2) *Sets $C(x)$ are given as in (4);*

3)

$$l \leq \begin{cases} \frac{\mu}{L}, & if L \geq 4\mu \\ \frac{2\mu}{L^2}\Big(L+\mu-\sqrt{\mu(2L+\mu)}, & if L < 4\mu \end{cases};$$

Then for a good choice of parameters $\lambda(t)$ and $\beta(t)$ of method, the trajectory is stabilized in the norm to the unique solution of quasi-variational inequalities (3).

Note that we also establish rate of convergence of the proposed method.

References

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