Multi-criteria design of control rules for nonlinear dynamic problems by using visualization of the Pareto frontier

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A method for supporting the multi-criteria designing the control rules (control synthesis) for the case of nonlinear dynamic systems with uncertainty is proposed. It is assumed that the dynamics of the system is described by the multi-step equation

$$x(t+1) = x(t) + f(x(t), u(t), z(t))$$

where $x(t)$ is the state vector at the time moment t, $u(t)$ is the control vector at the time moment t, $z(t)$ is the value of the stochastic variable at the time moment t, $t = 0, 1, \ldots$. The initial state vector $x(0)$ is given.

The dynamics of the system depends on both the control and the external impact $z(t)$, which is assumed to be a realization of a stochastic process with the discrete time. The problem of choosing the control rule is considered, i.e. one has to design an algorithm which relates the control $u(t)$ to the state $x(t)$. It is assumed that the control rule is periodical in time with the time-period τ, where the structure of the control rule and its period τ should be given by the experts. Namely,

$$u(t) = h(x(t), p(t), t)$$

where $p(t)$ is the vector of parameters at the time moments $t + k \cdot \tau$ while $t = 0, 1, \ldots, \tau - 1$ and k is any non-negative integer number. It is assumed that the parameter vector $p(t)$ must belong to a given set $P(t), t = 0, \ldots, \tau - 1$. The control problem with multiple criteria is considered, i.e. there are m conflicting criteria (objectives) used for selecting the decision rule. The criterion values can be computed by using the given vector-function

$$y = g(x(\cdot), u(\cdot), z(\cdot)).$$

An example of such type of problems is provided by designing the release rule for the reservoir of a hydropower plant (HPP). In this case the state is the water level in the reservoir, the control is the water release through the dam of the HPP in unit time, and the state equation is the water balance. The release rule relates the release to the water level, the time period of the year and, may be, some forecasted values (say, the volume of inflow into the reservoir for the next time period). The release rule must meet various requirements including the requirements on electricity production, the environmental requirements and the technical requirements as well as the requirement to ensure the needs of different water users and consumers. Thus, the task of deciding on the rule of water release is related to multiple decision criteria. In the framework of this application, the structure of dependence of the release on the water level is usually given by experts, but the parameters of the dependence must be found. The problem of designing the release rule for the reservoir of the HPP can be reduced to multi-criteria selecting the parameters of the given dependence in the case of uncertainty (see [1]).

Traditionally the water management expert reduce this problem to a decision problem without uncertainty. To do it, they use the historical information about the inflow to the reservoir, which is available in the form of a long-term historical hydrological series. Such information can be applied for the identification of the stochastic process of water inflow, which can be used then for generating synthetic hydrologic series of any length. Alternatively, such information can be used in the study directly. We do not dwell on this issue in details and assume that, in the framework of the problem of designing the rules of water release of the HPP reservoirs, a sufficiently long series of inflow values has already been prepared, and one can use it (jointly with the balance equation and formula for criteria evaluation) to compute the criterion values for the particular parameters of the control rules.

Exactly the same assumptions are made about the general mathematical problem of multi-objective control synthesis considered in this article. In the general case, we assume that a sufficiently long time series of the values of the stochastic variable $z(t), t = 0, \ldots, T-1$, is given, which can be considered as a representative realization of a stochastic process. The length of the time series must include a sufficiently great number of periods of the control rule τ. Using the series of the stochastic value $z(t)$ helps to reduce the problem to a multi-criteria problem with full information. Now, to describe the problem in the standard form of a multi-criteria decision problems, let us denote by w the collection of parameter vectors for all time moments of the period, i.e. $(p(0), \ldots, p(\tau-1))$. Let us denote by W the direct product of the sets $P(0), \ldots, P(\tau-1)$. Thus, the inclusion $w \in W$ must hold. For any particular collection of parameter vectors $w \in W$, the value of the criterion vector y can be computed by using the state equation, the time series of the values of the stochastic variable $z(t), t = 0, \ldots, T-1$, and the criteria formula. Let us denote this relation by $y = h(w)$. Let us assume that it is desirable to decrease the values of all m criteria. By this, we get the multi-criteria decision problem in the standard form

$$y \to min$$

while $y = h(w), w \in W$. We solve this multi-criteria problem by approximating and visualizing the Pareto frontier.

The process of approximating and visualizing the Pareto frontier is supported by the Interactive Decision Maps (IDM) technique, which we describe here in short (see [2] for details). Let consider the above multi-criteria decision problem. Let $Y = h(W)$ be the feasible criterion set. Since all criteria of the criterion vector y must be minimized, the criterion vector y dominates (in the Pareto sense) the criterion vector y' if, and only if, $y \leq y'$ and $y \neq y'$. Then, the Pareto frontier of the set Y is defined as

$$P(Y) := \{y \in Y : \{y' \in Y : y' \leq y, y' \neq y\} = \emptyset\}.$$

The IDM technique displays the Pareto frontier for more than two criteria by using the preliminary approximation of the Edgeworth-Pareto Hull (EPH) of the set Y, denoted by $H(Y)$. The set $H(Y)$ is introduced as follows. Let $\mathbf{R}^m_+$ be the non-negative orthant in $\mathbf{R}^m$, i.e.

$$\mathbf{R}^m_+ := \{y \in \mathbf{R}^m : y_i \geq 0, i = 1, \ldots, m\}.$$

Then, the EPH is given by

$$H(Y) = Y + \mathbf{R}^m_+.$$

It is important that $H(Y)$ is the largest set satisfying $P(H(Y)) = P(Y)$.

The Pareto frontier is displayed interactively through the display of bi-criterion slices of $H(Y)$ which are defined as follows. Let (y_1, y_2) designate two criteria, the so-called "axis criteria", and z denote the remaining criteria, which we shall fix at some values z^*. A bi-criterion slice $G(H(Y))$ of $H(Y)$, parallel to the criterion plane (y_1, y_2) and related to z^* is defined as

$$G(H(Y), z^*) = \{(y_1, y_2) : (y_1, y_2, z^*) \in H(Y)\}.$$

Note that a slice of $H(Y)$ contains all feasible combinations of the values for the "axis criteria" when the values of the other criteria are not worse than z^*. The bi-criterion slices of $H(Y)$ are used in the IDM technique to display *decision maps.* To define a particular decision map, the user has to specify the "third", or color-associated, criterion among the criteria from z. Then, a decision map is a collection of superimposed slices of $H(Y)$, for which the values of the color-associated criterion change, while the values of the remaining criteria are fixed. If one compares the slices of $H(Y)$ for two different values of the color-associated criterion, the slice for the worst value of this criterion encloses the slice for its better value. For this reason, efficient frontiers of slices (tradeoff or compromise curves) displayed at a decision map do not intersect.

Multiple theoretical and numerical studies as well as real-life applications of the IDM technique are related to the case of the convex set $H(Y)$ (see [2]). In this paper the problems with the nonconvex set $H(Y)$ are considered. In this case the approximation of the set $H(Y)$ is constructed in the form of the union of finite number of cones with vertices at the points of the set Y. More exactly, the approximation is given by a finite set T of points of the set Y, which are hopefully close to $P(Y)$. The set

$$T^* = T + \mathbf{R}^m_+$$

provides an internal approximation to the set $H(Y)$. Note that T^* is the Edgeworth-Pareto Hull of the set T. To construct a set T, which points are close to the Pareto frontier, one can use various methods. In particular, if the Lipschitz constants are known for the criteria functions, one can use methods based on the covering of the set W by parallelotops (see [3]). New kind of methods (hybrid methods) were developed recently ([4]). These methods were designed for the case, in which there is no information on the value (or even on the existence) of Lipschitz constants for the criteria functions. Thus, they can be used if the criteria functions are given by a black box, that is, a computational module of unknown structure, which only guarantees the computation of the criteria vector y for a given decision vector w. The hybrid methods combine random search, local optimization, contraction of the search domain, and genetic methods. Constructing of multiple two-dimensional cross-sections of the set T^* does not require large time. Thus, such approximation provides an opportunity of interactive visualizing the Pareto frontier.

Application of the hybrid methods for approximating the EPH and visualizing the Pareto frontier in the problem related to release rule design resulted in improving the criterion values in comparison to the existing release rule developed by experts.

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