Optimal control of dynamic process with boundary optimization under linear constraints

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Abstract

An optimal control problem with two boundary problems of finite convex optimization is considered. Dynamics of the controlled process is described by linear differential equations. We propose an iterative extragradient-type process, formulated in the functional subspace of piecewise differentiable trajectories. The convergence in controls and trajectories has been proved.

1 Problem statement

An optimal control problem with free right end is considered. The dynamics of the process is described by a system of linear equations

$$\frac{d}{dt}x(t) = D(t)x(t) + B(t)u(t), \quad t_0 \le t \le t_1, \quad (1)$$

where D, B are continuous $n \times n$ and $n \times r$ matrices, $u : [t_0, t_1] \to L_2^r$ is control, $x : [t_0, t_1] \to L_2^n$ are trajectories.

Controls are assumed to be piecewise continuous functions of the set

$$U = \{u \in L_2^r[t_0, t_1] |\ u_i \in [u_i^-, u_i^+], i = \overline{1, r}\}. \quad (2)$$

Let us introduce the linear subspace $L_2^n[t_0, t_1] \subset \check{C}^n[t_0, t_1]$ of piecewise differentiable functions, satisfying the initial condition x_0, and assume that $x, x^* \in \check{C}^n[t_0, t_1]$.

Adding to system (1) the initial condition x_0 and any control $u \in U$, and solving this problem, we can find x. When a control u "runs" over the whole set U, then the right ends x_1 of trajectories describe the set of attainability X_1, where the functional $\varphi_1(x_1)$ is defined. We also require that the right ends of trajectories are subject to additional restrictions $C_1x_1 \le c_1$.

Collecting together, we have the following optimal control problem with boundary optimization:

$$\begin{cases} \dfrac{d}{dt}x(t) = D(t)x(t) + B(t)u(t),\ t_0 \le t \le t_1, \\ \qquad\qquad x(t_0) = x_0,\ u(t) \in U, \\ x_1^* \in \text{Argmin}\{\varphi_1(x_1) |\ C_1x_1 \le c_1, x_1 \in X_1\}. \end{cases} \quad (3)$$

It is necessary for a given x_0 to find the optimal control u^* such that the right end x_1^* of corresponding trajectory x^* was minimizing (on the set X_1 under constraints $C_1x_1 \le c_1$) the functional $\varphi_1(x_1)$. We assume that the solution u^*, x^* exists, and, in general, not unique.

2 Reducing the problem to finding a saddle point of Lagrangian

It is known that the optimization problem using the Kuhn-Tucker theorem can be reduced to finding the saddle points of the Lagrangian [2]. In this case the Lagrangian has the form

$$L_1(p_1, \psi; x, u) = \varphi_1(x_1) + \langle p_1, C_1x_1 - c_1\rangle +$$

$$+ \int_{t_0}^{t_1} \langle \psi(t), D(t)x(t) + B(t)u(t) - \frac{d}{dt}x(t)\rangle dt,$$

and is defined for all $x \in \check{C}^n[t_0, t_1]$, $u \in U$, $p_1 \in R_+^m$, $\psi \in \check{C}_*^n[t_0, t_1]$, where $\check{C}_*^n$ is dual subspace. Its saddle point $(p_1^*, \psi^*; x^*, u^*)$ satisfies, by definition, the system

$$L_1(p_1,\psi;x^*,u^*) \le L_1(p_1^*,\psi^*;x^*,u^*) \le$$

$$\le L_1(p_1^*,\psi^*;x,u). \qquad (4)$$

The equivalence of the original problem (3) to the saddle problem (4) has been proved.

As a result of transformations, using the transition to the adjoint operator and the formula for integration by parts, the problem (4) has been reduced to a system that reflects the well-known Pontryagin's maximum principle:

$$\frac{d}{dt}x^*(t) = D(t)x^*(t) + B(t)u^*(t),\ x^*(t_0) = x_0^*, \qquad (5)$$

$$p_1^* = \pi_+(p_1^* + \alpha(C_1x_1^* - c_1)), \qquad (6)$$

$$\frac{d}{dt}\psi^*(t)+D^T(t)\psi^*(t) = 0,\ \psi_1^* = \nabla\varphi_1(x_1^*)+C_1^Tp_1^*, \qquad (7)$$

$$u^*(t) = \pi_U(u^*(t) - \alpha B^T(t)\psi^*(t)), \qquad (8)$$

where $\pi_+(\cdot)$ and $\pi_U(\cdot)$ – projection operators on R_+^m and U, $\alpha > 0$.

3 Method of solution

To solve the latter system it seems natural to apply the method of simple iteration:

$$\frac{d}{dt}x^k(t) = D(t)x^k(t) + B(t)u^k(t),\ x^k(t_0) = x_0^*, \qquad (9)$$

$$p_1^{k+1} = \pi_+(p_1^k + \alpha(C_1x_1^k - c_1)), \qquad (10)$$

$$\frac{d}{dt}\psi^k(t)+D^T(t)\psi^k(t) = 0,\ \psi_1^k = \nabla\varphi_1(x_1^k)+C_1^Tp_1^k, \qquad (11)$$

$$u^{k+1}(t) = \pi_U(u^k(t) - \alpha B^T(t)\psi^k(t)),\ k = 1,2,... \qquad (12)$$

The process of (9)–(12) is the simplest of the known numerical methods. For a strictly contractive mapping it converges at a geometric rate. But it is known that in the case of saddle problems, in general, such methods do not converge to the solution. So, to solve the system (5)–(8) the extragradient-type approach has been used as a controlled simple iteration method [1].

Each iteration of the extragradient method is divided into two half-steps:

1) prediction half-step

$$\frac{d}{dt}x^k(t) = D(t)x^k(t) + B(t)u^k(t),\ x_0^k = x_0^*, \qquad (13)$$

$$\bar{p}_1^k = \pi_+(p_1^k + \alpha(C_1x_1^k - c_1)), \qquad (14)$$

$$\frac{d}{dt}\psi^k(t)+D^T(t)\psi^k(t) = 0,\ \psi_1^k = \nabla\varphi_1(x_1^k)+C_1^T\bar{p}_1^k, \qquad (15)$$

$$\bar{u}^k(t) = \pi_U(u^k(t) - \alpha B^T(t)\psi^k(t)); \qquad (16)$$

2) basic half-step

$$\frac{d}{dt}\bar{x}^k(t) = D(t)\bar{x}^k(t) + B(t)\bar{u}^k(t),\ \bar{x}_0^k = x_0^*, \qquad (17)$$

$$p_1^{k+1} = \pi_+(p_1^k + \alpha(C_1\bar{x}_1^k - c_1)), \qquad (18)$$

$$\frac{d}{dt}\bar{\psi}^k(t)+D^T(t)\bar{\psi}^k(t) = 0,\ \bar{\psi}_1^k = \nabla\varphi_1(\bar{x}_1^k)+C_1^T\bar{p}_1^k, \qquad (19)$$

$$u^{k+1}(t) = \pi_U(u^k(t) - \alpha B^T(t)\bar{\psi}^k(t)),\ k = 1,2,... \qquad (20)$$

Note that in this method two systems of differential equations are solved at each half-step and two projection operations in the variables p_1 and u are fulfilled.

4 Proof of convergence for the method

The following theorem about the convergence of the method to solution of the problem has been proved.

Theorem 1. *Let the set of optimal trajectories of the problem* (5)-(8) *is not empty and belongs to subspace* $\check{C}^n[t_0,t_1]$, $\varphi_1(x_1)$ *is convex and differentiable function with gradient satisfying the Lipschitz condition with constant* K_1; $D(t), B(t)$ *are matrices being continuous on segment* $[t_0,t_1]$, U *is a set of admissible controls of the form* (2). *Then the sequence*

$$\{\|u^k - u^*\|^2_{L_2^r} + |p_1^k - p_1^*|^2_{R^m}\},$$

generated by iterative process (13)–(20), *where* $0 < \alpha < \frac{1}{\sqrt{2}K}$, $K^2 = (\|C_1\|^2+$

$B^2_{max}K^2_1\frac{1}{2D_{max}}\left(e^{2D_{max}t_1}-1)\right)t_1B^2_{max}E^{2D_{max}t_1}$, *decreases monotonically in* $L^r_2 \times R^m$. *At the same time any weakly convergent subsequence* $\{u^{k_i}(t)\}$ *converges to* $u^*(t)$*, and a corresponding subsequence* $\{x^{k_i}(t)\}$ *converges to* $x^*(t)$ *in the uniform norm* $\breve{C}^n[t_0,t_1]$.

If $\{u^k(t)\}$ *has a strong limit point then the process* $\{p^k_1, x^k(t), u^k(t), \psi^k(t)\}$ *converges in norm to the solution* $p^*_1, x^*(t),\ u^*(t), \psi^*(t)$*, moreover by* $u(t), p_1$ *– monotonically.*

5 Example of economic model of Harrod-Domar's type

As an economic interpretation of the above problem formulation we have considered a generalization of the well-known Harrod-Domar model of development of production [3]. This model relates the change in income from production with the consumption and investment relationship.

In the classical model, instead of (1) we have the only equation, where $x(t)$ is an income received by a company at the moment t. One part of income is consumed ($P(t)$) and the rest is invested in the future development of enterprises ($I(t)$): $x(t) = P(t) + I(t)$. Income growth is proportional to the investment: $\frac{d}{dt}x(t) = \frac{1}{K}x(t) - \frac{1}{K}P(t)$ (constant K characterizes the capital productivity, where K is a ratio of capital to income growth).

We consider this model as applied to the multidimensional situation. Suppose that $x(t)$ and $P(t)$ are continuous vector-functions, where $x_i(t)$, $i = \overline{1,n}$, is income earned from the sale of i-type products, $P_i(t)$ – costs, consumption (pay workers' wages, depreciation, etc.), $I_i(t)$ is investment in i-th kind of product. Capital-gains income is given by the matrix $K(t)$ and also depends continuously on time. We have a system of ODE:

$$\frac{d}{dt}x(t) = K^{-1}(t)I(t) - K^{-1}(t)P(t),$$

where $K_{ij}(t)$ is a coefficient characterizing the influence (equity) of investment in the j-th kind of product for the production of i-th kind of product.

It is required, using the regulation of consumption (which in our formulation acts as a control), to maximize revenue $x_1 = x(t_1)$ at $t = t_1$ with additional linear constraints $C_1x_1 \le c_1$. The meaning of this inequality: income is restricted in practice by finite resources and consumer demand. As the terminal function to be minimized, we can take

$$\varphi_1(x_1) = -\langle \hat{c}, x_1\rangle,$$

where $\hat{c}$ is vector of weights.

Drawing a parallel with (1), for this model we have: $D(t) = -B(t) = K^{-1}(t)$, $u(t) = P(t)$.

In particular, when $n = 1$ and there is no terminal problem, we have the classical model of Harrod-Domar. Note that the terminal problem does not necessarily have the kind of optimization problem. Some equation, reflecting, for example, a balance of economic situation, can also act as a terminal problem.

So, we have the generalization of the economic model of the Harrod-Domar: first, the classical model has been extended to the multidimensional case, second, terminal problem was added, and finally, the iterative extragradient method for solving the resulting problem has been proposed.

An important feature of the proposed formulation of the problem lies in the fact that the dynamics of production (ODE system) is consistent with the strategy (terminal problem). The real possibilities of the manufacturer also are taken into account (terminal constraints).

References

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