Source Conditions and Regularization of Ill-posed Quadratic Programming Problems

Ill-posedness of a minimization problem means that small perturbations of initial data of a problem may produce big changes in its set of solutions. One of the important moments in studying of an ill-posed problem is a construction of the regularizing methods for its solving.

1. We will deal with minimization problem

$$J(u) = \frac{1}{2}\|Au - f\|^2 \to \inf,\ u \in U, \qquad (1)$$

where $A : H \mapsto F$ is a bounded linear operator from Hilbert space H to Hilbert space F, $f \in F$ - a fixed element, and $U \subseteq H$ - a closed convex set, assuming that the sets U_* and U_∞ of the solutions of the given problem and of the corresponding problem without constraints

$$J(u) = \frac{1}{2}\|Au - f\|^2 \to \inf,\ u \in H, \qquad (2)$$

are nonempty. In this case the problems

$$\|u\|^2 \to \inf, u \in U_*,\ \|u\|^2 \to \inf, u \in U_\infty \qquad (3)$$

have unique solutions, that we will denote by u_* and u_∞ and call *normal solutions* of problems (1) and (2).

A prominent examples of the problems of the previous type, with infinite-dimensional spaces H and/or F can be found in [6].

2. Even in case of the absence of constraint $u \in U$, these problems can be ill-posed, i.e. it is possible that there is $\tilde{u}$ which is far from the set of solution, such that $\|Au - f\|^2 \approx J_* = \inf J(u) : u \in U\}$. In case of infinite-dimensional spaces H and F, the ill-posedness of the problem obviously comes from the fact that the range $R(A) := \{Au : u \in H\}$ of the operator A is non-closed. However, if the operator A is known only approximately, then this problem can be ill-posed even in case of $R(A) = \overline{R(A)}$. In this case, in order to solve the given problem, one has to use methods of regularization (see [6], [5]).

Usually the bounds of the accuracy of the regularization methods for solving ill-posed problems (1) and (2) were obtained for classes of problems defined by so-called source conditions concerning their normal solutions. The well known conditions of this type are *power source conditions* that were used widely in [5] for obtaining the estimates of the convergence rate of regularization methods for solving linear operator equations. This condition can be presented in the form

$$u_\infty = |A|^p h_*,\ \text{where } h_* \in H,\ |A|^p = (A^*A)^{\frac{p}{2}},\ p > 0, \qquad (4)$$

It seems quite natural having in mind that $u_\infty \in \overline{R(A^*)}$, where $R(A^*)$ is the range of the operator A and $\overline{R(A^*)}$ its closure in norm of the space H. Hence, the solution u_∞ is densely surrounded by the elements from $R(A^*)$.

Let us note that in several recent papers concerning *linear operator equations without constraints* there were considered so-called *general source conditions* [4]

However, the presence of the constraints notable complicates the procedure of regularization. In [2] was constructed an example which shows that *the rate of convergence, in dependent of the boundary of the set U, can be arbitrary slow.*

We will study an accuracy of the regularization methods on the class of the problems of type (1) with normal solutions that satisfy *projective source condition*:

$$u_* = \pi_U(|A|^p h_*),\ h_* \in H, p > 0. \qquad (5)$$

3. Suppose that the continuous functions $g_\alpha : [0,a] \mapsto R$, $a > 0$ satisfy the conditions:

$$1 - tg_\alpha(t) \geq 0,\ t \in [0,a],\ \|A\| \leq a, \qquad (6)$$

$$\frac{1}{1+\beta\alpha} \leq g_\alpha(t) \leq \frac{1}{\beta\alpha},\ t \in [0,a],\ \beta > 0, \qquad (7)$$

$$\sup_{0\leq t\leq a} t^p(1 - tg_\alpha(t)) \leq \gamma_p \alpha^p, \qquad (8)$$

$$\alpha > 0,\ \gamma_p = const,\ 0 \leq p \leq p_0,\ p_0 > 0. \qquad (9)$$

Let us note that the family of the functions $g_\alpha(t) = (t+\alpha)^{-1}$ and $g_\alpha(t) = t^{-1}(1-(1+t)^{-m})$ (that defines Tikhonov methods of regularization and its iterated variants) satisfy these conditions.

As an approximation of the normal solution u_* of problem (1) can be taken the unique solution u_α of the variational inequality

$$\left\langle g_\alpha^{-1}(A^*A)u_\alpha - A^*f_\delta, u - u_\alpha \right\rangle \geq 0,\ \forall u \in U, \qquad (10)$$

In case of Tikhonov regularization ($g_\alpha^{-1} = t + \alpha$), variational inequality (**??**) becomes

$$\left\langle (A_\eta^* A_\eta + \alpha I)u_\alpha - A_\eta^* f_\delta, u - u_\alpha \right\rangle \geq 0,\ \forall u \in U.$$

The following theorem is an example of the statements related to convergence of the methods of regularization.

Theorem 1 *Suppose conditions (6)-(9) are satisfied.*

(a) If the parameter α in (10) converges to 0 *then $u_\alpha \to u_*$.*

(b) If condition (5) is satisfied then

$$\|u_\alpha - u_*\| \leq const \cdot \alpha^{\frac{p}{p+2}},\ 0 \leq p \leq 2p_0 - 1. \qquad (11)$$

References

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