Cutting Plane Method for Clique Inequalities in the Facility Location Problem with clients' preferences *

T. V. Gruzdeva†, X. B. Klimentova‡

Institute for System Dynamics and Control Theory SB RAS,
†gruzdeva@icc.ru, ‡Xenia.Klimentova@icc.ru

1 Introduction

The facility location problem with clients' preferences (FLPCP) is a special case of the bilevel facility location problem [1, 2, 10]. At the upper level, the supplier chooses a subset of facilities to be open. Then, at the lower level, each client chooses one of these facilities according to the client's own preferences. The problem is to choose a set of facilities to be open at the upper level so that the clients are served with minimal total cost. FLPCP is also known as facility location problem with order [3].

This problem was firstly considered in [5]. To find an optimal solution of FLPCP a reduction to integer linear programs is usually used. Lower bounds of LP-relaxation for these integer programs usually used to improve exact methods, based on Brunch and Cut schemes. The lower bounds for FLPCP were studied in [1, 3, 6, 9, 10]. In [3], the lower bound with some families of valid inequalities was proposed and the computational experiment was carried out. The new family of valid inequalities was proposed in [10] too. They are based on the extended formulation of FLPCP from [1, 6]. The computational experience with these inequalities was performed in [9] and approved the effectiveness of the new lower bound in comparison with the other approaches [1, 3]. Moreover, in [10] the new formulation and the corresponding lower bound for FLPCP was designed with clique inequalities which were constructed on the basis of relaxation of FLPCP to the Set Packing problem. But the computational experiment with clique inequalities was not performed.

The purpose of this paper is checking the efficiency of the family of clique inequalities [10]. We implemented a cutting plane method for calculating the corresponding lower bound. A computational experiment was carried out on series of test instances and approved the efficiency of the proposed formulation of FLPCP.

2 Problem statement

In FLPCP we are given by $I = \{1, \dots, m\}$ the set of facilities; $J = \{1, \dots, n\}$ the set of clients; $f_i \geq 0$ $(i \in I)$ is the cost of opening the facility i; $c_{ij} \geq 0$ $(i \in I,\ j \in J)$ is the matrix of the production and delivery costs for servicing the clients; $g_{ij} \geq 0$ $(i \in I,\ j \in J)$ is the matrix of the clients' preferences, more precisely, if $g_{i_1 j} < g_{i_2 j},\ i_1 \neq i_2$, then client j prefers the open facility i_1 to the open facility i_2.

The following variables are used in the model: $x_{ij} = 1$ if the client j is served from facility i (otherwise $x_{ij} = 0$); $y_i = 1$ if facility i is open (otherwise, $y_i = 0$).

Using this notations, one can write the following bilevel integer programming formulation [1, 10]:

$$\begin{gathered} \sum_{i \in I} \sum_{j \in J} c_{ij} x_{ij} + \sum_{i \in I} f_i y_i \ \downarrow\ \min_{y}, \\ y_i \in \{0, 1\}, \quad i \in I, \end{gathered} \tag{1}$$

where $x(y)$ is optimal solution of the client prob-

*The investigation was carried out under financial support of RFBR (the grant number 11-01-00270-a)

lem:

$$\begin{aligned} &\sum_{i\in I}\sum_{j\in J} g_{ij}x_{ij} \downarrow \min_{x}, \\ &\sum_{i\in I} x_{ij} = 1, \quad 0 \le x_{ij} \le y_i, \\ &x_{ij} \in \{0,1\}, \quad i \in I, j \in J. \end{aligned} \tag{2}$$

The objective function of the upper level problem (1) minimizes the cost of servicing clients and opening facilities. The objective function of the lower level problem (2) guarantees that the clients are served by the most preferable facility.

If the clients' optimal choice in problem (2) is unique, then bilevel integer problem (1)—(2) can be reduced to equivalent Linear Integer Program [1, 10].

Let's denote by $W_{ij} \stackrel{\triangle}{=} \{k \in I \mid g_{kj} < g_{ij}\}$ the set of facilities which are worse for the client j than the facility i and $B_{ij} \stackrel{\triangle}{=} \{k \in I \mid g_{kj} > g_{ij}\}$ the set of facilities which are better for the customer j than the facility i. Linear Integer Programming formulation for the FLPCP is following:

$$\sum_{i\in I}\sum_{j\in J} c_{ij}x_{ij} + \sum_{i\in I} f_i y_i \downarrow \min_{x,y}, \tag{3}$$

$$y_i + \sum_{k\in W_{ij}} x_{kj} \le 1, \quad i \in I, j \in J, \tag{4}$$

$$\sum_{i\in I} x_{ij} = 1, \quad j \in J, \tag{5}$$

$$0 \le x_{ij} \le y_i \le 1, \quad i \in I, j \in J, \tag{6}$$

$$x_{ij}, y_i \in \{0,1\}, \quad i \in I, j \in J. \tag{7}$$

The constraints (4) ensure, that each client is served by the most preferable facility. In the next section, we describe techniques for strengthening formulation (3)–(7) which were proposed in [10].

3 Clique Inequalities

The relation between the (3)–(7) and the well-known Set Packing problem [7] was examined in [10]. The connection was exploited to derive valid inequalities for FLPCP.

Consider a $0-1$ matrix D and a nonnegative vector d. The formulation for the Set Packing problem with the variables z is:

$$\langle d, z\rangle \uparrow \max_{z}, \quad Dz \le 1.$$

A Set Packing problem can be easily transformed into a Stable Set problem by considering graph $G = (V, E)$ constructed as follows. Every column of D is associated with a vertex in G. The vertices i and j are connected by an edge if and only if the columns i and j are not orthogonal.

We will denote by P_G the polytope of the Set Packing problem, which is the convex hull of $0-1$ vectors corresponding to stable sets of graph G. Any complete subgraph in a given graph is called *clique*. A clique that is not a subclique of a larger one is called a *locally maximal clique*. Let K be a clique in G. It is known (see [8]) that the clique inequality $\sum_{k\in K} z_k \le 1$ is valid for P_G and it is facet defining if K is a locally maximal clique.

It was noted in [10] that initial constraints (4), (5) and valid inequalities from [3] and [10] define the Set Packing relaxation of the FLPCP (for more information see [10]) Denote by $\mathcal{K}$ the set of all cliques in graph G constructed as described above for this relaxation of the FLPCP, and consider the following family of clique inequalities:

$$\sum_{k\in K_i} (x,y)_k \le 1, \quad K_i \in \mathcal{K}. \tag{8}$$

These inequalities (8) are valid for the polytope of FLPCP (see [10]).

Now we develop the Cutting Plane method for calculating the lower bound, corresponding to the formulation of the FLPCP with clique inequalities. To find locally maximal cliques in graph G we use the continuous formulation of clique problem as a convex quadratic minimization problem over the canonical simplex with a constraint given by difference of two convex functions. This formulation was proposed in [4]:

$$\left.\begin{aligned} \phi(x) \stackrel{\triangle}{=} \sum_{i=1}^{N} \frac{1}{w_i} x_i^2 \downarrow \min, \quad x \in S, \\ \Phi(x) \stackrel{\triangle}{=} \left\langle x, \overline{B}x \right\rangle \le 0, \end{aligned}\right\} \tag{9}$$

where $S \stackrel{\triangle}{=} \{x \in I\!R^N \mid x \ge 0, \sum_{i=1}^{N} x_i = 1\}$, node's weights are given by the vector $w \in \mathcal{R}_+^{|V|}$ and the matrix $\overline{B} = \{\overline{b_{ij}}\}$ $(N \times N)$ is constructed by following rule: $\overline{b_{ij}} = \dfrac{1}{2w_i} + \dfrac{1}{2w_j}$ if $i \neq j$, $(i,j) \notin E$; $\overline{b_{ij}} = 0$ otherwise.

The characteristic vector $z(K,w) = (z_1,..,z_N)$ of a clique K is: $z_i = \dfrac{w_i}{W(K)}$ if $i \in K$, $z_i = 0$ otherwise, where $W(K) = \sum_{i \in K} w_i$.

It was proved in [4] that $K_* \subset V$ is a maximum weighted clique of $G(V,E)$ iff $z(K,w)$ is a global solution of problem (9). An algorithm for finding maximum weighted cliques was developed in [4] as well. It's based on global algorithm for problem (9). The algorithm was tested on the DIMACS benchmark graphs and showed promising results. In this paper we adopt this algorithm for finding violated clique inequalities (8) in our Cutting plane method. The scheme of the method is given below.

Cutting plane method for clique inequalities

Step 0. Solve linear program (3)–(6). Let $(\bar{x}, \bar{y})$ be it's solution.
Step 1. Construct graph $G(V,E)$ and weight vector $w := \bar{z} \triangleq (\bar{x}, \bar{y})$.
Step 2. Using the algorithm for finding maximum weighted cliques from [4] construct the set of violated clique inequalities: $Q \subset \mathcal{K}$ such that $W(K) > 1 \ \forall K \in Q$.
Step 3. If $Q = \emptyset$, then **STOP**: the lower bound of FLPCP is $\sum_{i \in I} \sum_{j \in J} c_{ij}\bar{x}_{ij} + \sum_{i \in I} f_i \bar{y}_i$.
Step 4. Using set Q construct clique inequalities (8) and add them to the formulation of FLPCP.
Step 5. Solve linear program (3)–(6), with the current subset of inequalities (8) and goto Step 1.

4 Numerical results

We carried out computational experiment with the proposed cutting plane method for inequalities family (8). The obtained by CP lower bound for the optimal value of FLPCP was compared with the other known lower bounds from [3] and [9]. We used the test field from [3] for the experiment. This test field consists of 3 groups of instances: "small" ones with $m = 50$ and $n = 50$; the second block of "middle" size problems, $m = 50$ and $n = 75$; and the the "large" problems with $m = 75$ and $n = 100$. To solve auxiliary linear programming and quadratic programming problems we used solvers IBM ILOG CPLEX (URL: www.ibm.com) and FICO Xpress Opitmization Suite (URL: www.fico.com). The computations were performed on PC Pentium-4 (3 GHz).

In tables 1-3 we present the integrality gaps for the 3 formulations of FLPCP, where $gap \triangleq \frac{Opt-LB}{Opt}100\%$, LB is the lower bound and Opt is the optimal value. The following notations are used in the tables: gap is the integrality gap for the lower bound of LP-relaxation of the initial fomulation (3)-(7), gap[3] is the gap for lower bound presented in [3], gap[9] is the gap obtained by cutting plane method in [9] for the inequalities family proposed in [10], finaly, gap_{new} is the inegrality gap for the lower bound obtained by implemented in this paper cutting plane method for clique inequalities.

Table 1. Integrality gaps for the problems $m = 50$, $n = 50$.

Name	*gap*	*gap* [3]	*gap*[9]	gap_{new}
132-1	10.27	8.66	1.95	**0**
132-2	14.44	11.82	4.49	**1.3**
132-3	11.97	10.10	3.55	**0**
132-4	6.80	6.07	1.61	**0**
133-1	9.52	8.47	0.28	**0**
133-2	6.15	5.25	0.47	**0**
133-3	12.61	11.73	2.97	**0**
133-4	7.60	6.30	1.64	**0.61**
134-1	12.12	7.79	1.99	**0**
134-2	7.12	5.43	0.00	**0**
134-3	12.66	12.23	4.30	**1.61**
134-4	13.25	11.84	5.58	**3.14**

As it can be seen in Table 1 the designed cutting plane method doesn't not manage to solve only 4 "small" problems of 12. For this instances the best gap[9] is significantly reduced.

The computational results for the problems of "middle" and "large" sizes are presented in Tables 2 and 3 correspondingly. The improvement of the lower bound for these groups is also significant. For the problems of the second class, the integrality gaps are reduced on 65%, 60% and 35% corespondingly for *gap*, *gap* [3] and *gap*[9]. As to problems of a "large" size the reduction of integrality gaps is 40%, 34% and 9%.

Table 2. Integrality gaps for the problems $m = 50$, $n = 75$.

Name	*gap*	*gap* [3]	*gap*[9]	gap_{new}
a75-50-1	27.67	24.47	16.47	**12.69**
a75-50-2	27.22	23.71	16.38	**12.76**
a75-50-3	27.14	23.66	16.08	**12.75**
a75-50-4	24.29	21.75	14.53	**11.36**
b75-50-1	28.11	24.49	13.73	**8.66**
b75-50-2	31.06	27.15	17.73	**12.09**
b75-50-3	28.22	24.36	14.58	**8.39**
b75-50-4	27.28	22.67	12.01	**6.78**
c75-50-1	31.85	26.83	14.75	**9.12**
c75-50-2	29.04	25.18	12.99	**7.22**
c75-50-3	28.47	22.29	11.19	**4.95**
c75-50-4	30.47	26.63	16.05	**10.23**

Table 3. Integrality gaps for the problems $m = 75$, $n = 100$.

Name	*gap*	*gap* [3]	*gap*[9]	gap_{new}
a100-75-1	21.27	18.96	11.99	**10.55**
a100-75-2	27.19	25.20	18.77	**16.83**
a100-75-3	25.97	24.16	17.71	**15.48**
a100-75-4	24.57	22.27	15.98	**14.79**
b100-75-1	31.80	28.51	21.64	**20.55**
b100-75-2	33.08	30.48	23.29	**21.74**
b100-75-3	34.47	30.97	23.76	**22.41**
b100-75-4	28.56	27.07	19.18	**17.72**
c100-75-1	32.15	27.95	20.52	**18.82**
c100-75-2	31.49	28.35	20.14	**17.87**
c100-75-3	32.30	28.92	20.78	**18.25**
c100-75-4	32.36	29.41	21.16	**19.11**

Thus, the results of the computational experiment confirm the efficiency of the implemented cutting plane method for the family of clique inequalities in view of improving the lower bounds.

References

[1] E. V. Alekseeva and Yu. A. Kochetov, *Genetic Local Search for the p-Median Problem with Clients Preferences.* Diskret. Anal. Issled. Oper, 2007. V. 14, N. 1. P. 3-31 [in Russian].

[2] V. L. Beresnev, *Discrete Facility Location Problems and Polynomials of Boolean Variables.* Inst. Mat., Novosibirsk, 2005 [in Russian].

[3] L. Cánovas, S. García, M. Labbé, and A. Marín, *A strengthened formulation for the simple plant location problem with order.* Operations Research Letters, 2007. V. 35, N 2. P. 141–150.

[4] T. V. Gruzdeva, *On solving the maximum weighted clique problem via d.c. constraint problem.* Proceedings of the Toulouse global optimization workshop, 2010. P. 59–62.

[5] P. Hanjoul and D. Peeters, *A facility location problem with clients' preference orderings.* Regional Science and Urban Economics, 1987. V. 17. P. 451–473.

[6] P. Hansen, Y. Kochetov, and N. Mladenović, *Lower bounds for the uncapacitated facility location problem with user preferences.* Les Charies du GERAD G-2004-24, 2004.

[7] G. L. Nemhauser and L. A. Wolsey, *A recursive procedure to generate all cuts for 0-1 mixed integer programs.* Mathematical Programming, 1990. V. 46. P. 379–390.

[8] M. W. Padberg, *On the Facial Structure of the Set Packing Polyhedra.* Mathematical Programing, 1973. V. 5. P. 199-215.

[9] I. L. Vasilyev and K. B. Klimentova, *The Branch and Cut Method for the Facility Location Problem with Clients Preferences.* Journal of Applied and Industrial Mathematics, 2010. V. 4, N. 3. P. 441-454.

[10] I. L. Vasilev, K. B. Klimentova, and Yu. A. Kochetov, *New Lower Bounds for the Facility Location Problem with Clients Preferences.* Computational Mathematics and Mathematical Physics, 2009. V. 49, N. 6. P. 1010-1020