On modifications of finite non-cooperative games with a convex structure

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Let X_i be a nonempty subset of a Euclidean space $\mathbf{E}_i$, f_i a function taking finite real values and defined over the set X, which is the direct product of the subsets $X_1, \dots, X_k$, $1 \le i \le k$, and $\mathbf{E}$ the direct product of the Euclidean spaces $\mathbf{E}_1, \dots, \mathbf{E}_k$, $k \ge 2$. Introduce a non-cooperative game Γ with k players by defining a set of strategies X_i and a payoff function f_i for each player i, $1 \le i \le k$.

We say that the game Γ is *convex* if the function f_i, $1 \le i \le k$, is continuous on its domain $\{x = (x_1, \dots, x_k)\colon\ x_s \in X_s,\ 1 \le s \le k,\ x_i \in \widetilde{X}_i\}$ and concave with respect to $x_i \in \widetilde{X}_i$ for any fixed $x_s \in X_s$, $1 \le s \le k$, $s \ne i$, where $\widetilde{X}_i$ is a convex open set comprising X_i. Let $X^*(\Gamma)$ be the set of all Nash equilibrium points of the game Γ. If Γ is a convex game then the set of its Nash equilibrum points is non-empty: $X^*(\Gamma) \ne \emptyset$. However, the convexity of the game Γ does not generally imply that the set $X^*(\Gamma)$ is convex, too.

The problem of finding the Nash equilibrium points of the game Γ can be reduced to the solution of a special variational inequality related to Γ. Let T be a point-to-set mapping that associates a subset $T(x)$ of the Euclidean space $\mathbf{E}$ with each point $x \in X$. The variational inequality in question defined by the mapping T has the form,

$$t \in T(x), \quad \langle t, x' - x\rangle \ge 0 \quad \forall \quad x' \in X. \tag{1}$$

Denote the set of all solutions to the variational inequality (1) by $\overline{X}^*(T)$.

In order to reduce the Nash equilibrium problem of the convex game Γ to the solution of the variational inequality (1), we associate the game Γ with a point-to-set mapping T_Γ defined by the following relationship

$$T_\Gamma(x) = \{t = (t_1, \dots, t_k)\colon -t_i \in \partial_{x_i} f_i(x),\ 1 \le i \le k\}, \tag{2}$$

$x \in X$, where $\partial_{x_i} f_i(x)$ is the superdifferential of the function f_i with respect to x_i calculated at the point $x \in X$.

The definitions of the Nash equilibrium and the solution of the variational inequality (1) imply that the set $X^*(\Gamma)$ of the Nash equilibrium points of the game Γ coincides with the set $\overline{X}^*(T_\Gamma)$ of the solutions to the variational inequality (1) if $T = T_\Gamma$. Therefore, the methods solving the variational inequality (1) defined by the mapping T_Γ can be used to find the Nash equilibrium points of the non-cooperative game Γ. In [1, 2], we described a sufficiently efficient numerical method to solve the variational inequality (1), which, under some additional assumptions about the mapping T, generates sequences of iterations converging to the set $\overline{X}^*(T)$.

We say that a convex non-cooperative game Γ has a *convex structure* if the mapping T_Γ defined by (2) is monotone. If the game Γ has a convex structure then the mapping $T = T_\Gamma$ satisfies the requirements guaranteeing the convergence of the above-mentioned numerical method to solve the variational inequality (1).

Now consider a modification $\Gamma(R)$ of the convex game Γ defined by the following payoff functions of the players:

$$f_i(x, R_i) = f_i(x) - \frac{R_i}{2}\|x_i\|^2, \quad x \in X, \quad 1 \le i \le k, \tag{3}$$

where $R = (R_1, \dots, R_k) \ge 0$, and $f_i(x)$ $(x \in X)$ is the payoff function of player i in the game Γ.

It is interesting to find out for what values of the vector parameter R the convex game $\Gamma(R)$ has a convex structure. This talk answers this question for finite non-cooperative games Γ with mixed strategies.

Each player has a finite number of strategies in a finite non-cooperative game. To describe such games, it is worthy to make use of tables, every entry of which is numerated with several indices (namely, k indices, according to the number of players). Such tables are sometimes referred to as *k-dimensional* ones. We denote a k-dimensional table A by $A = (a_{s_1 s_2 \dots s_k})_{n_1 n_2 \dots n_k}$, where s_α is the α-th index running the integers from 1 to n_α. Here, $1 \le \alpha \le k$, and $a_{s_1 s_2 \dots s_k}$ is the entry of the table A defined with k indices taking values $s_1, s_2, \dots, s_k$, respectively.

Now let us examine a finite non-cooperative game with k players, in which player i governs n_i startegies, while her payoff function is determined by a k-dimensional table $A_i = (a^{(i)}_{s_1 s_2 \ldots s_k})_{n_1 n_2 \ldots n_k}$, where $a^{(i)}_{s_1 s_2 \ldots s_k}$ is the payoff to player i if player α selects strategy s_α, $1 \le \alpha \le k$. If we expand the players' strategy sets by allowing mixed strategies, then we come to a game Γ with k players, in which

$$\begin{aligned} X_i &= \Big\{ x_i = (x_{i1}, \ldots, x_{in_i})\colon \sum_{j=1}^{n_1} x_{ij} = 1, \\ &\quad x_{ij} \ge 0,\ j = 1, \ldots, n_i \Big\}, \\ f_i(x) &= \sum_{s_1 \ldots s_k} a^{(i)}_{s_1 \ldots s_k} x_{1s_1} \cdots x_{ks_k},\ 1 \le i \le k, \\ &\quad x = (x_1, \ldots, x_k) \in X = X_1 \times \cdots \times X_k. \end{aligned} \tag{4}$$

Necessary and sufficient requirements toward the tables A_i such that the finite game Γ defined with these tables has a convex structure, were described in the previous papers [3, 4]. These conditions make it clear that the subclass of finite games with a convex structure is extremely narrow. For instance, when $k = 2$, a convex structure is found only in such finite bi-matrix games, for which the aggregate players' payoffs matrix $A = A_1 + A_2$ can be represented in the following form:

$$A = (a_{s_1 s_2})_{n_1 n_2} = (\alpha'_{s_1} + \alpha''_{s_2})_{n_1 n_2}. \tag{5}$$

If a bi-matrix game Γ defined by the matrices $A_i = (a^{(i)}_{s_1 s_2})_{n_1 n_2}$, $i = 1, 2$, satisfies condition (5), then it is easy to check that its set of Nash equilibrium points coincides with the set of saddle points of the antagonistic game Γ of two players (the matrix game) defined by the matrix $\overline{A}_1 = (\overline{a}^{(1)}_{s_1 s_2})_{n_1 n_2} = (a^{(1)}_{s_1 s_2} - \alpha''_{s_2})_{n_1 n_2}$. Due to this fact, we will refer to a bi-matrix game satisfying (5) as an *almost-matrix* game. Hence, the class of bi-matrix games with a convex structure consists of the almost-matrix games and thus does not practically differ from the class of matrix games.

Now let $A = (a_{ij})_{nm}$ be an arbitrary matrix with m rows and n columns, $u_i = \sum_{j=1}^n a_{ij}/n$, $v_j = \sum_{i=1}^m a_{ij}/m$, $\overline{a} = \sum_{i=1}^m \sum_{j=1}^n a_{ij}/(mn)$. Denote

$$r(A) = \Big(\sum_{i=1}^m \sum_{j=1}^n (a_{ij} - u_i - v_j + \overline{a})^2 \Big)^{1/2}. \tag{6}$$

As it is demonstrated in [3], the nonnegative quantity $r(A)$ can be interpreted as the degree of the best Euclidean approximation of a matrix A by the matrices of the form $(\alpha'_i + \alpha''_j)_{mn}$.

Consider a game $\Gamma = \Gamma(A_1, \ldots, A_k)$ defined, according to (4), by the collection of tables A_i, $1 \le i \le k$. Due to (3), we will refer to the game $\Gamma(R) = \Gamma(A_1, \ldots, A_k, R)$, in which player i has the payoff function

$$\begin{aligned} f_i(x_1, \ldots, x_k, R_i) &= \sum_{s_1 \ldots s_k} a^{(i)}_{s_1 \ldots s_k} x_{is_1} \cdots x_{ks_k} \\ &- \frac{R_i}{2} \|x_i\|^2,\ x_i = (x_{i1}, \ldots, x_{in_i}) \in X_i,\ 1 \le i \le k, \end{aligned} \tag{7}$$

as a *modification* of the game Γ; here, $R = (R_1, \ldots, R_k) \ge 0$ is the vector parameter of the modification.

For each pair (i, j), $i \ne j$, of players of a finite game Γ, we understand by $s(i, j)$ a collection of (pure) strategies of the remaining $k - 2$ players. Denote the set of all such collections by $S(i, j)$. It is evident that $S(i, j)$ is contains $\prod_{1 \le \alpha < k,\, \alpha \ne i,j} n_\alpha$ elements. We denote by $\Gamma(s(i, j))$ a bi-matrix game generated by the game Γ, which comprises players i, j from the game Γ, whereas the remaining $k - 2$ players have fixed pure strategies $s(i, j) \in S(i, j)$. Let $A_{ij}(s(i, j))$ be the aggregate players' payoff matrix in the bi-matrix game $\Gamma(s(i, j))$. For arbitrary i, j ($1 \le i \le k$, $1 \le j \le k$, $i \ne j$), define

$$r_{ij} = \max_{s(i,j) \in S(i,j)} r(A_{ij}(s(i, j))), \tag{8}$$

where $r(A)$ for any matrix A is determined by relationship (6). Clearly, $r_{ij} = r_{ji}$.

Theorem 1. *Let $\Gamma = \Gamma(A_1, \ldots, A_k)$ be a finite non-cooperative k player game, $k \ge 2$, defined by (4) with the tables A_i, and $\Gamma(R) = \Gamma(A_1, \ldots, A_k, R)$ be a modification of the game Γ, defined by (7), $R^*_i = 0,5 \sum_{1 \le j \le k,\, j \ne i} r_{ij}$, where the numbers r_{ij} are given by formula (8), $1 \le i \le k$, $R = (R_1, \ldots, R_k)$, $R^* = (R^*_1, \ldots, R^*_k)$. Then the modified game $\Gamma(R)$ has a convex structure if $R \ge R^*$.*

The latter result is demonstrated for $k = 2$ in [3], whereas the case $k > 2$ is examined in [5]. Making use of theorem 1 one can establish another necessary and sufficient condition guaranteeing the existence of a convex structure in a finite non-cooperative game $\Gamma = \Gamma(A_1, \ldots, A_k)$.

Theorem 2. *A game $\Gamma = \Gamma(A_1, \ldots, A_k)$ has a convex structure if, and only if for any $i \ne j$, $1 \le i \le k$, $1 \le j \le k$ and $s(i, j) \in S(i, j)$, the bi-matrix game $\Gamma(s(i, j))$ is an almost-matrix game.*

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