Space-Covering Technique for Gobal Multiobjective Optimization

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1 Introduction

Many numerical methods for solving multiobjective optimization problems have been proposed so far. Most of them are heuristic, i.e. they don't guarantee the optimality of the found solutions. In this paper we extend the non-uniform space covering technique for the global multiobjective optimization. This technique was initially proposed in [1] for global optimization of non-linear functions and later [2] applied to multiobjective problems. This method constructs the finite set of feasible points and proves its global ε-optimality. The space-covering algorithm is guaranteed to converge to a global ε-Pareto set within a specified tolerance in a finite number of steps.

A multiobjective optimization problem is stated as follows:

$$\min_{x \in X} f(x). \tag{1}$$

Function $f(\cdot) : \mathbb{R}^n \to \mathbb{R}^m$ is a vector objective comprising m scalar objectives. Assume that $f(\cdot)$ is continuous and a feasible set $X \subseteq \mathbb{R}^n$ is compact. The image $\Omega = f(X)$ of the feasible set is called an *attainable set*.

Let $\mathrm{SW}(\theta) = \{y \in \mathbb{R}^m : y \leq \theta\}$ and $\mathrm{NE}(\theta) = \{y \in \mathbb{R}^m : y \geq \theta\}$. For an arbitrary set $\Theta \subseteq \mathbb{R}^m$ denote its *Pareto-optimal subset* as $\mathcal{P}(\Theta) = \{\theta \in \Theta : \Theta \cap \mathrm{SW}(\theta) = \theta\}$. The goal of multiobjective optimization is to find a Pareto set $\Omega_* = \mathcal{P}(\Omega)$ and a Pareto-optimal solution set $X_* \subseteq X$ such that $f(X_*) = \Omega_*$. Notice that the Pareto set is searched in $\mathbb{R}^m$ and a solution is searched in $\mathbb{R}^n$.

2 ε-Pareto set

Except for special cases where the Pareto set is finite or representable by a finite collection of faces of polyhedron it is in general very difficult to determine the entire Pareto set. Therefore the suitable approximation concept is needed.

Let $\varepsilon > 0$ be a positive real number. Following [2] we say that a finite set $\Omega_\varepsilon \subseteq \Omega$ is an ε-*Pareto set* if

1. for any $\omega \in \Omega_*$ there exists such $\omega_\varepsilon \in \Omega_\varepsilon$ that $\omega_\varepsilon - \varepsilon \cdot e_m \leq \omega$;
2. $\mathcal{P}(\Omega_\varepsilon) = \Omega_\varepsilon$.

Here $e_m = (1, 1, ..., 1)$ is a vector with all components equal to 1.

From the definition of the Pareto-optimal subset it immediately follows that for any $\omega \in \Omega$ there exists such $\omega_\varepsilon \in \Omega_\varepsilon$ that $\omega_\varepsilon - \varepsilon \cdot e_m \leq \omega$. With other words: each point in the attainable set is ε-dominated by at least one point from the ε-Pareto set.

A set $A_\varepsilon \subseteq X$ such that $f(A_\varepsilon) = \Omega_\varepsilon$ is called an ε-*optimal solution* of the problem (1).

The following statement establishes a relation between Pareto and ε-Pareto sets. Let

$$\rho(a, B) = \inf_{b \in B} \|a - b\|$$

be a *distance between a point* $a \in \mathbb{R}^m$ *and a set* $B \subseteq \mathbb{R}^m$ and

$$d(A, B) = \sup_{a \in A} \rho(a, B)$$

be a *distance between sets* $A \subseteq \mathbb{R}^m$ *and* $B \subseteq \mathbb{R}^m$.

Statement 1. Let Ω be compact. Then for any $\delta > 0$ there exists such $\varepsilon > 0$, that for any ε-Pareto set Ω_ε the following inequality holds

$$d(\Omega_*, \Omega_\varepsilon) \leq \delta, \tag{2}$$

Statement 1 describes the limit behaviour but doesn't provide a link between ε and δ. Such connection can be established for Geoffrion's points. Recall the slightly modified definition from [3]:

A point $\hat{\omega} \in \Omega$ is called *Geoffrion's point* if there is a real number $M > 0$ such that for all $i = 1, ..., m$ and $\omega \in \Omega$ satisfying $\omega^{(i)} < \hat{\omega}^{(i)}$ there exists an index j such that $\hat{\omega}^{(j)} < \omega^{(j)}$ and

$$\frac{\hat{\omega}^{(i)} - \omega^{(i)}}{\omega^{(j)} - \hat{\omega}^{(j)}} \leq M.$$

Statement 2. If ω is a Geoffrion's point then for any ε-Pareto set Ω_ε, $\varepsilon > 0$ the following inequality holds

$$\rho(\omega, \Omega_\varepsilon) \leq \varepsilon\sqrt{m} \cdot \max(1, M). \quad (3)$$

3 Space Covering Technique

The first version of space-covering method proposed in [2] was based on an assumption that objectives satisfy the Lipschitzian condition and relied on Lipschitzian lower bounds. We extend this algorithm to the case of arbitrary underestimations.

In the sequel we'll use the notion of the *Lebesque set.* For arbitrary sets $\Lambda \subseteq \Omega$, $Z \subseteq \mathbb{R}^n$ and a function $f(\cdot) : \mathbb{R}^n \to \mathbb{R}^m$ define a *Lebesque set* $\mathcal{L}(f(\cdot), Z, \Lambda) = \{x \in Z : f(x) \in \mathrm{NE}(\Lambda)\}$, where $\mathrm{NE}(\Lambda) = \cup_{\lambda \in \Lambda} \mathrm{NE}(\lambda)$.

Consider a sequence $X_1, \dots, X_k$, $X_i \subseteq X$ and a sequence $\Lambda_1, \dots, \Lambda_k$, $\Lambda_i \subseteq \Omega$. Sets Λ_i are finite. Let $\mu_i(\cdot)$ be a lower bound for $f(\cdot)$ over X_i i.e. $\mu_i(x) \leq f(x)$ for any $x \in X_i$. The *covering sequence* consists of sets $\mathcal{L}_1, \dots, \mathcal{L}_k$ satisfying the following property:

$$\mathcal{L}_i \subseteq \mathcal{L}(\mu_i(\cdot), X_i, \Lambda_i - \varepsilon), i = 1, \dots, k,$$

where $\Lambda_i - \varepsilon = \{x : x = \lambda - \varepsilon \cdot e_m \text{ for some } \lambda \in \Lambda_i\}$.

Theorem 1. If $\cup_{i=1}^k \mathcal{L}_i = X$ then $\Omega_k = \mathcal{P}\left(\cup_{i=1}^k \Lambda_i\right)$ is an ε-Pareto set.

The goal of the covering method is to construct the set Ω_k and the collection of sets $\{\mathcal{L}_i\}$ satisfying the premises of Theorem 1. Below we outline the basic form of this method. This method uses the **Update** procedure that updates an archive containing the list of non-dominated solutions with a new point:

procedure Update (A, x)
Parameters:
A — current archive;
x — new point;

1. For each point a in A do:
   if $f(a) \leq f(x)$ then return;
   else if $f(x) \leq f(a)$ then remove a from A;
2. Add x to A.

To save the computing time the objective values are stored together with points and are not reevaluated at each iteration in a loop.

Covering algorithm

1. Initialize a list $S = \{X\}$ and an archive $A = \emptyset$.
2. Take a set X_i from S.
3. Take a point $x_i \in X_i$ and update the archive A: Update(A, x_i).
4. Compute $\alpha_i = \min_{x \in X_i} \mu_i(x)$.
5. If $\omega - \varepsilon \cdot e_m \leq \alpha_i$ for at least one point ω from $f(A)$ then exclude X_i from the list S and go to step 2, otherwise partition X_i into two or more smaller subsets and replace the set X_i in the list S by the generated sequence.
6. If list S is empty then terminate the algorithm, otherwise go to step 2.

By applying Theorem 1 one can easily show that after termination the archive contains the ε-solution A and the corresponding ε-Pareto set $f(A)$.

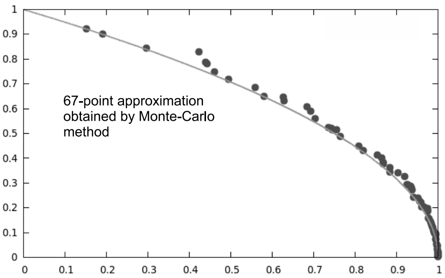
The outlined version of the covering algorithm can be efficiently implemented for a simple set X that allows easy partitioning. Simplest examples of such sets are boxes and polytopes.

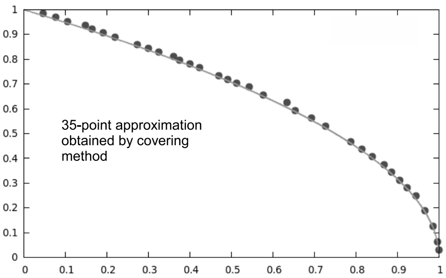
4 Implementation

The covering algorithm for multiobjective optimization has been implemented in the BNB-Solver framework [4]. The BNB-Solver is a generic framework for implementing optimization algorithms on serial and parallel computers. Computational experiments for various test problems demonstrated that this algorithm reliably constructed the ε-Pareto set approximations in a reasonable time. Obtained approximations provide uniform covering of the Pareto-set. To illustrate our approach we consider a simple bi-criteria problem:

$$f_1(x) = (x_1 - 1)x_2^2 + 1, f_2(x) = x_2, x_1 \in [0, 1], x_2 \in [0, 1].$$

The following figure demonstrates the difference between the Pareto set approximation obtained by the Monte-Carlo method (upper plot) and ε-Pareto set with $\varepsilon = 0.05$ computed by the covering method (lower plot). Both methods didn't use local search. The covering method clearly produced a better approximation with a less number of points (35 vs 67).





Similar to the single objective case the multiobjective version of the covering method significantly benefits from local search techniques. Numerical experiments showed that the approximations obtained with the help of the local search are more accurate and the number of algorithm steps is much less with respect to the pure covering approach. At the moment we use a simple local search method that descends along the feasible direction that minimises all objectives locally (if possible). More elaborate techniques were proposed in papers [6, 7].

We extended the proposed method to constrained multiobjective problems. In [8] it was shown that a non-uniform covering approach provides an opportunity to solve non-linear optimization problems where some (or all) variables are discrete. This property can be exploited for multiobjective optimization as well. Due to the space limitations we do not describe the multiobjective optimization with functional and integrality constraints here.

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