OPTIMAL STABILIZATION OF MULTIPLY CONNECTED CONTROLLED SYSTEMS

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It is known that for nonlinear controlled systems a number of methods of stabilization on the first approach of the steady and unsteady motions are developed. Critical cases are allocated and ways of a finding of stabilizing control in critical cases are developed [1], [2].

In this paper we suggest a method for solving the problem of optimal stabilization with reference to multiply connected nonlinear controlled dynamic systems given in the form of nonlinear systems of the ordinary differential equations. The decision of the specified problem reduces to the solving of a problem of optimal stabilization in the sense of V.V. Rumyantsev [3].

The method of solving the problem of optimal stabilization applies to multiply connected systems with homogeneous principal parts. Thereby critical cases are investigated also.

We consider multiply connected nonlinear controlled dynamic system

$$\frac{dx_s}{dt} = f_s(t,x_s)+F_s(t,x)+B_s(t,x_s)u_s \equiv \Phi_s(t,x,u),$$
$$s=\overline{1,q}, \qquad (1)$$

where $x_s \in R^{n_s}$, $u_s \in R^{r_s}$, and vector functions $f_s(t,x_s)$, $F_s(t,x)$ and matrixes $B_s(t,x_s)$ of dimension $n_s \times r_s$ are defined in domain

$$\Omega = \{t,x : t \geq t_0, ||x|| < h, 0 < h = \text{const}\}. \qquad (2)$$

For system (1) it is accepted that

$$x = \left(x_1^T,...,x_q^T\right)^T, \; R^{n_1}\oplus...\oplus R^{n_q} = R^n,$$
$$\Phi_s(t,0,0) \equiv 0,$$
$$R^{r_1}\oplus...\oplus R^{r_q} = R^r, \qquad s=\overline{1,q}.$$

We assume that for systems

$$\frac{dx_s}{dt} = f_s(t,x_s), \quad s=\overline{1,q}, \qquad (3)$$

in domain (2) there are continuously differentiable functions of Lyapunov $v_s(t,x_s)$, $s=\overline{1,q}$, satisfying the Lyapunov theorem on the uniformly asymptotic stability. It should be noted that this assumption is not burdensome because the control action, by analogy with the stabilization in critical cases can be written as

$$u_s(t,x) = u_s^{\text{loc}}(t,x_s) + u_s^{\text{glob}}(t,x),$$

where $u_s^{\text{loc}}(t,x_s)$ is the control action at the level of subsystems and $u_s^{\text{glob}}(t,x_s)$ is the control action at the level of original system. For the first approximation system (3) with respect to (1) there exists a Lyapunov function $v(t,x) = \sum\limits_{s=1}^{q} v_s(t,x_s)$, where $v_s(t,x_s)$ are definite positive Lyapunov functions for the individual subsystems of (3) admitting an infinitesimal upper limit, and the total time derivatives along the solutions of corresponding subsystems are negative definite functions. Therefore, we assume that the above Lyapunov functions are known. Under these assumptions the problem of optimal stabilization applied to systems (1) corresponds to the problem of optimal stabilization in the sense of V.V. Rumyantsev [3]. The difference is that we are considering the problem of optimal stabilization with regard to a term $F_s(t,x)$.

For the purpose of the solving of a problem of optimal stabilization on the basis of Lyapunov function $v(t,x) = \sum\limits_{s=1}^{q} v_s\,(t,x_s)$ we introduce a function of the form

$$B[v;t,x,u] = \sum_{s=1}^{q}\Big[\frac{\partial v_s(t,x_s)}{\partial t} + (\nabla v_s(t,x_s))^T \times$$

$$\times (f_s(t,x_s) + F_s(t,x) + B_s(t,x_s)u_s +$$

$$+ u_s^T \beta_s(x_s)u)\Big] + \Psi(t,x). \tag{4}$$

Here $\nabla v_s(t,x_s) = \mathrm{col}\Big(\frac{\partial v_s}{\partial x_1}, \ldots, \frac{\partial v_s}{\partial x_{q_s}}\Big)$ and $\beta_s(x_s)$ are positive definite symmetric matrixes. According to Rumyantsev and Krasovsky theorems [3]–[5] for optimal control and optimal Lyapunov function we obtain

$$B[v^0;t,x,u^0] = 0,$$

and at all other control functions we have

$$B[v^0;t,x,u] \geq 0.$$

Consequently, optimal control we will find from the system

$$\frac{\partial B}{\partial u_s} = (\nabla v_s(t,x_s))^T B_s(t,x_s) + \frac{1}{2}\beta_s(x_s)u_s,$$

$$s = \overline{1,q}. \tag{5}$$

According to (4), (5) we will have

$$u_s^0(t,x_s) = -\beta_s^{-1}(x_s)\nabla v_s(t,x_s)^T B_s(t,x_s),$$

$$s = \overline{1,q}. \tag{6}$$

Substituting $u_s^0(t,x_s)$ in (5) we obtain the algebraic equation with respect to function $\Phi(t,s)$ in the form

$$B[v^0;t,x,u^0] = 0,$$

from which we define a function

$$\Psi(t,x) = \sum_{s=1}^{q}\Big(w_s(t,x_s) + u_s^{0^T}\beta_s(t,x_s)u_s^0 -$$

$$- (\nabla v_s(t,x_s))^T F_s(t,x)\Big). \tag{7}$$

Here functions $w_s(t,x_s) = \left.\frac{dv_s}{dt}\right|_{(3)}$ are definitely negative. Thus, if the function $\Psi(t,x)$ is positive definite the quality functional of control finally becomes

$$J(u^0) = \int\limits_{t_0}^{\infty}\Big(\sum_{s=1}^{q}\Big(-w_s(t,x_s) + u_s^{0^T}\beta_s(t,x_s)u_s^0 -$$

$$- \left(\mathrm{grad}_{x_s} v_s(t,x_s), F_s(t,x)\right) +$$

$$+ u_s^T\beta_s(t,x_s)u_s\Big)\Big)dt. \tag{8}$$

Thus, if for systems (3) we know single-valued continuous positive definite Lyapunov functions $v_s(t,x_s)$, $s = \overline{1,q}$, allowing an infinitesimal limit and having negative definite full derivatives on solutions of subsystems (3) and if function (7) is definite positive, then the function $v(t,x) = \sum\limits_{s=1}^{q} v_s(t,x_s)$ is the optimal Lyapunov function for system (1) optimized by control (6) with to functional (8).

It is known that for nonlinear systems of differential equations of general form the conditions of theorems on asymptotic stability on the first nonlinear approach are hardly verified. However, in some cases nonlinear systems of differential equations is possible to search for fairly easily verifiable conditions under which we prove the asymptotic stability of the equilibrium of the first nonlinear approach. Nonlinear systems right-hand sides are homogeneous (generalized homogeneous) vector functions of the order $\mu > 1$ have been studied in [6]–[8], which shows the theorems about asymptotic stability on the first nonlinear approach with the conditions which are easily verified. These results lead to the solution of problems of optimal stabilization of nonlinear control multiply connected systems of the form

$$\frac{dx_s}{dt} = X_s^{(\mu_s)}(x_s) + \sum_{j=1}^{q} R_{sj}(t,x) + B_s(x_s)u_s \equiv$$

$$\equiv \Phi_s(t,x,u). \tag{9}$$

Here $x_s \in R^{n_s}$, $x = \Big(x_1^T, ..., x_q^T\Big)$, $X_s^{(\mu_s)}(x_s)$ are homogeneous of order $\mu_s > 1$ continuously differentiable vector functions. Continuous functions

$R_{sjj}(t,x)$ are defined in domain (2) and for these functions conditions

$$||R_{sj}(t,x)|| \le c_{sj}||x_1||^{\alpha_{sj}^{(1)}} \dots ||x_q||^{\alpha_{sj}^{(q)}},$$

$$c_{sj} \ge 0,\ \alpha_{sj}^{(i)} \ge 0,$$

are hold. We assume that $\Phi_s(t,0,0) \equiv 0$, $s = \overline{1,q}$, and that equilibrium states of systems

$$\frac{dx_s}{dt} = f_s(x_s), \quad s = \overline{1,q},$$

are asymptotically stable. As a Lyapunov function for system (9) in this case we consider the function

$$v_s(x) = \sum_{s=1}^{q} v_s(x_s),$$

where $v_s(x_s)$ are Lyapunov functions for systems (10) satisfying the conditions: (i) $v_s(x_s)$ and $w_s(x_s)$ are positive definite functions; (ii) $v_s(x_s)$ and $w_s(x_s)$ are gomogeniously positive functions of order $m_s+1-\mu_s$ and m_s, where m_s are enough large rational numbers with odd denominator and even numerator; (iii) functions $v_s(x_s)$ are continuously differentiable and

$$(\nabla v_s)^T f_s(x_s) = -w_s(x_s).$$

For the system (9) the problem of optimal stabilization has a unique solution in closed form. The function $B[v;t,x,u]$ and quality functional $J(u^0)$ will have correspondent forms. Thus, for system (9) we can apply the above arguments for (1), but with the functions of a particular species.

On the basis of the presented results with applications of results of works [9]–[11] algorithms of optimal stabilization of multiply connected nonlinear controlled systems are developed.

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