Modified Lagrange function method for solving the terminal two-person game in optimal control

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Problem statement

At a fixed time interval $[t_0, t_1]$ the dynamic analogue of finite two-person game with a Nash equilibrium is considered. The game is determined by the linear controlled differential systems with free left (x_{10}, x_{20}) and right (x_{11}, x_{21}) ends.

The initial condition at the left end is defined as the solution of the problem of convex programming. If controls $u_1(t), u_2(t)$ "run"on the sets U_1, U_2, then the left and right endpoints of the controlled trajectories $x_1(t), x_2(t)$ describe their sets of attainability $X_1(t_0), X_2(t_0)$, $X_1(t_1), X_2(t_1)$.

On attainability sets the parametric convex programming problems are formulated, each of which describes the behavior of the first and second players. On the direct (Cartesian) product of attainability sets (on the left and right ends of the interval), these problems generate a finite two-person game with a Nash equilibrium.

The dynamic two-person game is formulated as follows: to find controls $u_1^*(t), u_2^*(t)$ and the corresponding trajectories $x_1^*(t), x_2^*(t)$, left (x_{10}^*, x_{20}^*) and right (x_{11}^*, x_{21}^*) ends of which are a Nash equilibrium two-person games, i.e., solutions of finite two-person games on the left and right ends of the interval $[t_0, t_1]$.

A formal statement of the problem has the form:

the first player

$$x_{10}^* \in \text{Argmin}\{f_1(x_{10}, x_{20}^*) + \varphi_1(x_{10}),\ C_{10}x_{10} = c_{10},\quad x_{10} \geq 0, x_{10} \in X_1(t_0)\}, \tag{1}$$

$$\frac{d}{dt}x_1(t) = D_1(t)x_1(t) + B_1(t)u_1(t),\ x_{10}^* \in X_1(t_0), \tag{2}$$

$$U_1 = \{u_1(t) \in L_2^r[t_0, t_1] |\ u_1(t) \in [u_1^-, u_1^+]\},\quad t_0 \leq t \leq t_1, \tag{3}$$

$$x_{11}^* \in \text{Argmin}\{f_1(x_{11}, x_{21}^*) + \varphi_1(x_{11}),\ C_{11}x_{11} = c_{11},\quad x_{11} \geq 0, x_{11} \in X_1(t_1)\}, \tag{4}$$

the second player

$$x_{20}^* \in \text{Argmin}\{f_2(x_{10}^*, x_{20}) + \varphi_2(x_{20}),\ C_{20}x_{20} = c_{20},\quad x_{20} \geq 0, x_{20} \in X_2(t_0)\}, \tag{5}$$

$$\frac{d}{dt}x_2(t) = D_2(t)x_2(t) + B_2(t)u_2(t),\quad x_{20}^* \in X_2(t_0), \tag{6}$$

$$U_2 = \{u_2(t) \in L_2^r[t_0, t_1] |\ u_2(t) \in [u_2^-, u_2^+]\},\quad t_0 \leq t \leq t_1, \tag{7}$$

$$x_{21}^* \in \text{Argmin}\{f_2(x_{11}^*, x_{21}) + \varphi_2(x_{21}), C_{21}x_{21} = c_{21},\quad x_{21} \geq 0, x_{21} \in X_2(t_1)\}, \tag{8}$$

where $X_1(t_0), X_1(t_1) \subset R^n, X_2(t_0), X_2(t_1) \subset R^n$. Here the pair of parametric convex programming problems (1),(5) together represent the initial two-person game with a Nash equilibrium. The solution to this game, the pair x_{10}^*, x_{20}^* (Nash solution), is the initial condition for a pair of differential equations (2),(6). The main purpose of the game subsystem (1),(5) is to define the initial conditions for another game subsystem

(2)-(4), (6)-(8). Both game subsystems are relatively independent: the first provides the initial conditions for the second one.

Note that here the payoff functions describe the overall interest of each player: $\varphi_1(x_{10}), \varphi_1(x_{11})$, $\varphi_2(x_{20}), \varphi_2(x_{21})$ are interests on which the players are not going to make concessions; $f_1(x_{10}, x_{20}), f_1(x_{11}, x_{21})$, $f_2(x_{10}, x_{20}), f_2(x_{11}, x_{21})$ - interests on which players make concessions to find a compromise.

The second subsystem (2)-(4) and (6)-(8) includes the controlled dynamics for the first and second players, and the terminal two-person game with a Nash equilibrium (4), (8). In this subsystem a pair of controls $u_1(t), u_2(t) \in U_1 \times U_2$ by means of differential equations (2),(6) generates a pair of trajectories $x_1(t), x_2(t) \in X_1(t) \times X_2(t)$. Right ends of the trajectories describe the attainability sets $X_1(t_1) \in R^n$ and $X_2(t_1) \in R^n$. On the direct product of these sets a pair of payoff functions $f_1(x_{11}, x_{21}) + \varphi_1(x_{11})$, $f_2(x_{11}, x_{21}) + \varphi_2(x_{21})$ is defined, and for each variable its own constraints polytope is given.

The whole subsystem (2)-(4),(6)-(8) defines in functional space a two-person game with a Nash equilibrium, more precisely, its generalization to functional infinite-dimensional spaces. Indeed, if in the system dynamics is absent, then the pair (4),(8) is a classic two-person game with a Nash equilibrium in finite-dimensional space (this game is an exact analog of (1),(5). If functions $f_i(x_{1i}, x_{2i}), i = 1, 2$, are not available in the original system, then the system becomes a pair of unrelated problems of optimal control. In this case, the interests of a pair of players are connected only by payoff functions and are not associated by dynamics. It is assumed that the solution $x_i^*(t), u_i^*(t)$ of game system under consideration exists.

1 Reduction to the problem of computing the fixed point of an extremal mapping

System (1)-(8), in general, is rather cumbersome structure. In order not to deal with each problem separately, we introduce new macro-variables and make the system scalar. First, let's do this for the subsystem (1),(5), after then for the subsystem (2)-(4), (6)-(8).

First note that the objective functions of problems (1),(5) for fixed values of the parameters $x_{20} = x_{20}^*, x_{10} = x_{10}^*$ are defined with respect to different variables. These variables describe the sets in different spaces. If we take the Cartesian (direct) product of these sets and consider on this square sum of the two objective functions, we can see that the problem of minimizing the sum of functions on the square will be equivalent to minimizing each function separately in its variable.

Denote

$$w^*(t)|_{t=t_0} = v^*(t)|_{t=0} = v^*(t_0) = v_0^*,$$

$$w^*(t)|_{t=t_1} = v^*(t)|_{t=1} = v^*(t_1) = v_1^*,$$

$$v_0 = \begin{pmatrix} x_1(t_0) \\ x_2(t_0) \end{pmatrix} = \begin{pmatrix} x_{10} \\ x_{20} \end{pmatrix},$$

$$v_1 = \begin{pmatrix} x_1(t_1) \\ x_2(t_1) \end{pmatrix} = \begin{pmatrix} x_{11} \\ x_{21} \end{pmatrix}, \; c_i = \begin{pmatrix} c_{1i} \\ c_{2i} \end{pmatrix},$$

$$D(t) = \begin{pmatrix} D_1(t) & 0 \\ 0 & D_2(t) \end{pmatrix},$$

$$B(t) = \begin{pmatrix} B_1(t) & 0 \\ 0 & B_2(t) \end{pmatrix},$$

$$C_0 = \begin{pmatrix} C_{10} & 0 \\ 0 & C_{20} \end{pmatrix}, \; C_1 = \begin{pmatrix} C_{11} & 0 \\ 0 & C_{21} \end{pmatrix}.$$

This assumes that all dimensions of the matrices and vectors are compatible. We write the system (1)-(4) and (5)-(8) compactly, then come to the problem of computing the fixed point $(v_0^*, v_1^*) \in W_0 \times W_1$ of extremal mapping.

$$v_0^* \in \text{Argmin}\{\Phi_0(v_0^*, w_0) + \varphi_0(w_0)|$$

$$C_0 w_0 = c_0, w_0 \geq 0, w_0 \in W_0 \subset R^{2n}\}, \qquad (9)$$

$$\frac{d}{dt} w(t) = D(t)w(t) + B(t)u(t),$$

$$w(t_0) = v_0^*, \ u(t) \in U, \qquad (10)$$

$$v_1^* \in \text{Argmin}\{\Phi_1(v_1^*, w_1) + \varphi_1(w_1)|$$

$$C_1 w_1 = c_1, \ w_1 \geq 0, w_1 \in W_1 \subset R^{2n}\}, \qquad (11)$$

where $t_0 \leq t \leq t_1$.

System (9)-(11) is the problem of computing the fixed points of extremal mappings on the set of initial conditions and the set of attainability. For fixed value of $v_0 = v_0^*$ the pair of tasks (9) and (10),(11) becomes a pair of convex programming problems in finite and infinite dimensional spaces.

2 The method of modified Lagrangian

It is assumed that the problems (9),(11) are sufficiently regular, and for finite-dimensional problem together with its infinite-dimensional analogue the Kuhn-Tucker theorem is holds. The latter means that there are vectors of Lagrange multipliers (dual solution) $p_0^*, p_1^* \in R^{2n}, \psi^*(t) \in \check{C}_*^n[t_0, t_1]$, such that the pairs of vectors (p_0^*, w_0), (p_1^*, w_1), $\psi^*(t), w^*(t), u^*(t)$ are saddle points of the Lagrangian

$$L_0(v_0^*, p_0, w_0) = \Phi_0(v_0^*, w_0) + \varphi_0(w_0) + \langle p_0, C_0 w_0 - c_0 \rangle \qquad (12)$$

for problems (9),

and for the Lagrangian

$$L_1(v_1^*, p_1, w_1, \psi(t), w(t)) =$$

$$= \Phi_1(v_1^*, w_1) + \varphi_1(w_1) + \langle p_1, C_1 w_1 - c_1 \rangle +$$

$$+ \int_{t_0}^{t_1} \langle \psi(t), D(t)w(t) + B(t)u(t) - \frac{d}{dt} w(t) \rangle dt \qquad (13)$$

of problem (10),(11) for all $w_1 \geq 0, p_1 \in R^{2n}$, $\psi(t) \in \check{C}_*^n[t_0, t_1], w(t) \in \check{C}^n[t_0, t_1], u(t) \in U, w(t_0) = v_0^*$, where p_0, p_1 и $\psi(t)$ are dual variables generated by the constraints of problems (9)-(11), where $v_0^*, v_1^*, w^*(t), u^*(t)$ are primal solutions of system (9)-(11).

Along with the Lagrangians (12) and (13) we introduce the modified Lagrangians

$$M_0(v_0^*, p_0, w_0) = \Phi_0(v_0^*, w_0) + \varphi_0(w_0) +$$

$$+ \frac{1}{2k} |p_0 + k(C_0 w_0 - c_0)|^2 - \frac{1}{2k} |p_0|^2 \qquad (14)$$

for all $w_0 \geq 0, p_0 \in R^{2n}$ of problem (9)

and

$$M(v_1^*, p_1, w_1, \psi(t), w(t)) =$$

$$= \Phi_1(v_1^*, w_1) + \varphi_1(w_1) +$$

$$+ \frac{1}{2k} |p_1 + k(C_1 w_1 - c_1)|^2 - \frac{1}{2k} |p_1|^2 +$$

$$+ \frac{1}{2k} \int_{t_0}^{t_1} |\psi(t) + k(D(t)w(t) + B(t)u(t) -$$

$$- \frac{d}{dt} w(t)|^2 dt - \frac{1}{2k} \int_{t_0}^{t_1} |\psi(t)|^2 dt \qquad (15)$$

for all $w_1 \geq 0, p_1 \in R^{2n}$, $\psi(t) \in \check{C}_*^n[t_0, t_1], w(t) \in \check{C}^n[t_0, t_1], u(t) \in U, w(t_0) = v_0^*$ of problem (10),(11), where p_0, p_1 and $\psi(t)$ are dual variables generated by the constraints of problems (10),(11).

Saddle point of the modified Lagrangian (15) is also a saddle point of Lagrangian (13) and is the solution of the original problem (10),(11) (see [1]). Similarly, it is true for (12),(14). Saddle condition for our function $M(v_1, p_1, w_1, \psi(t), w(t))$ we write in the form

$$v_1^*, v^*(t), u^*(t) \in$$

$$\text{Argmin}\{\Phi_1(v_1^*, w_1) + \varphi_1(w_1) +$$

$$+ \frac{1}{2k} |p_1^* + k(C_1 w_1 - c_1)|^2 - \frac{1}{2k} |p_1^*|^2 +$$

$$+ \frac{1}{2k} \int_{t_0}^{t_1} |\psi^*(t) + k(D(t)w(t) + B(t)u(t) -$$

$$- \frac{d}{dt} w(t)|^2 dt - \frac{1}{2k} \int_{t_0}^{t_1} |\psi^*(t)|^2 dt \mid w_1, w(t), u(t)\}, \qquad (16)$$

$$p_1^* = p_1^* + k(C_1 v_1^* - c_1), \qquad (17)$$

$$\psi^*(t) = \psi^*(t) + k(D(t)v^*(t) + B(t)u^*(t) - \frac{d}{dt} v^*(t). \qquad (18)$$

Using the system (16)-(18) by analogy with finite-dimensional case we write down the method of modified Lagrangian. In our infinite-dimensional case, it has the form [2]

$$v_1^{n+1}, v^{n+1}(t), u^{n+1}(t) \in \text{Argmin}\{\Phi_1(v_1^n, w_1)+$$

$$+\varphi_1(w_1) + \frac{1}{2k}|p_1^n + k(C_1 w_1 - c_1)|^2 - \frac{1}{2k}|p_1^*|^2+$$

$$+\frac{1}{2k}\int_{t_0}^{t_1} |\psi^n(t) + k(D(t)w(t) + B(t)u(t)-$$

$$-\frac{d}{dt}w(t)|^2 dt - \frac{1}{2k}\int_{t_0}^{t_1} |\psi^n(t)|^2 dt \mid w_1, w(t), u(t)\}, \quad (19)$$

$$p_1^{n+1} = p_1^n + k(C_1 v_1^{n+1} - c_1), \quad (20)$$

$$\psi^{n+1}(t) = \psi^n(t)+$$

$$+k(D(t)v^{n+1}(t) + B(t)u^{n+1}(t) - \frac{d}{dt}v^{n+1}(t). \quad (21)$$

Process (19)-(21) converges monotonically in norm on direct product of variables (controls, trajectories and variables of terminal problems) to a solution of the original problem.

Theorem 1. *If the set of solutions (9)-(11) is not empty and belongs to the subspace* $\check{C}^n[0,T] \subset L_2^n[0,T]$*, the functions* $\Phi_i(v_i^*, w_i) + \varphi_i(w_i), i = 1, 2$*, are positive semidefinite, and convex in the variables* w_i*, differentiable with respect to these variables, whose gradients satisfy the Lipschitz conditions, then the sequence of approximations generated by the process of (19)-(21) with the choice of the parameter* α *from the condition* $0 < \alpha < \alpha_0$*, decreases monotonically in the norm on direct product of variables (controls, trajectories and variables of terminal problems). At the same time, any weakly converging subsequence of controls* $u^{n_i}(t)$ *weakly converges to the optimal control* $u^*(t)$*, and a corresponding subsequence of trajectories* $v^{n_i}(t)$ *converges to optimal trajectory* $v^*(t)$ *in the uniform norm* $C^n[0,T]$*.*

If the sequence of controls $u^n(t)$ *has a strong limit point in the norm of* L_2^n*, then the process* $(v^n(t), u^n(t))$ *converges to a solution* $(v^*(t), u^*(t))$ *monotonically in norm of spaces* $L_2^n \times L_2^r$*.*

In the process of realization of the method (19)-(21) the vector of initial conditions v_0^* is used, which is first necessary to calculate by solving the equilibrium problem (9). The equilibrium problem can be solved by the same method (19)-(21), which in relation to the problem (9) has the form

$$v_0^{n+1} \in \text{Argmin}\{\Phi_0(v_0^*, w_0) + \varphi_0(w_0)+$$

$$+\frac{1}{2k}|p_0^n + k(C_0 w_0 - c_0)|^2 - \frac{1}{2k}|p_0^n|^2 \mid w_0 \geq 0, w_0 \in W_0\}$$

$$p_0^{n+1} = p_0^n + k(C_0 v_0^{n+1} - c_0)).$$

This process is a special case of (19)-(21), and its convergence to the solution follows from the above theorem.

Список литературы

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