ZONAL FEEDBACK CONTROL PROBLEMS FOR NON-LINEAR DYNAMIC SYSTEMS

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1. Introduction

Feedback control problems have been investigated by a lot of authors. The interest to this class of problems has increased for the last years due to the development of technical, computational, and measuring facilities of monitoring and of controlling technical and technological objects. They generally considered linear systems, and in case of non-linear systems corresponding linearized systems were used [1-3].

In the work, we investigate a class of feedback control problems for dynamic, in the general case, non-linear objects with lumped parameters. Necessary optimality conditions obtained and formulae for the gradient of the target functional on the optimized parameters allow to build algorithms of numerical solution to the feedback control problems. These algorithms are based on first order optimization methods. In the work, results of numerical experiments obtained when solving some test problems are given.

2. Problem statement

Let the controlled process be described by the following non-linear system of differential equations:

$$\dot{x}(t) = f(x(t), u(t), y), \quad t \in (0, T], \tag{1}$$

$$x(0) = x^0 \in X^0 \subset R^n, \quad y \in Y \subset R^m, \tag{2}$$

where $x(t)$ is n-dimensional vector-function defining the current state of the process; $u(t) \in U$ r-dimensional vector-function of control; $U \subset R^r$ the closed set of admissible values of the control actions; y m-dimensional vector of constant parameters of the process, the values of which are uncertain, but we have the admissible set of their values and density (weighting) function $\rho_Y(y) \geq 0$ defined on Y; X^0 the set of possible values of initial states of the process with given density (weighting) function $\rho_{X^0}(x^0) \geq 0$.

The problem consists in determining the control actions minimizing the following functional

$$J(u) = \int\limits_{X^0} \int\limits_{Y} I\left(u; x^0, y\right) \rho_{X^0}\left(x^0\right) \rho_Y(y)\, dY dX^0, \tag{3}$$

$$I\left(u; x^0, y\right) = \int\limits_0^T g(x(t), u(t))\, dt + \Phi(x(T)), \tag{4}$$

where $x(t) = x(t; x^0, y, u)$ is the solution to system (1) under admissible control $u(t)$, initial condition x^0, and values of the parameters of the process y. Functional (3), (4) defines the quality of the control optimal on the average on $x^0 \in X^0$ $y \in Y$. Control problems under uncertain information on the initial state and on parameters of the process, as well as some problems of control of cluster of trajectories are reduced to problem (1)-(4) [4].

Control of process (1) is realized taking into account the presence of feedback under which the state vector $x(t)$ is measured in full or in part. The measurements can be taken at given discrete points of time or continuously.

In the work, to control the process in the presence of feedback, we propose to choose the values of the control actions according to which given subset (zone) of the phase space the phase state belongs to.

Denote by $X \subset R^n$ the set of all possible states of the process under different admissible initial states $x^0 \in X^0$ and the values of parameters $y \in Y$, and of controls $u(t) \in U$ when $t \in [0,T]$.

Let the set X be partitioned into given number L of open subsets (zones) X^i such that

$$\overset{L}{\underset{i=1}{U}} \overline{X}^i = X, \ X^j \cap X^i = \emptyset,$$

$$i \neq j, \ \ i,j = 1,2,...,L,$$

where $\overline{X}^i$ is the closure of set X^i.

Problem A1. We have points of time $\tau_j \in [0,T]$, $j = 0,1,...,N$,$\tau_0 = 0$, at which it is possible to observe the current state of the process $x(\tau_j) \in X$. The frequency of the observations is such that when the state of the process is in some zone, it is observed at least once.

The values of control $u(t)$ when $t \in [\tau_j, \tau_{j+1})$ are assigned according to the value of the last observed current state of the process, namely, according to the set of the phase space which the measured (observed) current state belongs to:

$$u(t) = v^i = const, \ x(\tau_j) \in X^i, \ t \in [\tau_j, \tau_{j+1}), \tag{5}$$

$$v^i \in U \subset R^r, \ \ i = 1,2,...L, \ \ j = 0,1,...N-1,$$

assuming that $\tau_N = T$. It is required to determine the values v^i, $i = 1,2,...L$, optimizing functional (3).

Problem A2. The control actions are determined according to the results of observations at given discrete points of time $\tau_i \in [0,T]$, $i = 0,1,...,N$ in the form of linear functions of the measured values of the parameters of the process:

$$u(t) = K_1^i \cdot x(\tau_j) + K_2^i, \ \ x(\tau_j) \in X^i, \tag{6}$$

$$t \in [\tau_j, \tau_{j+1}), \ i = 1,2,...,N, \ j = 0,1,...,N-1,$$

Here K_1^i is the matrix of dimension $r \times n$; K_2^i r-dimensional vector; K_1^i, K_2^i are constant when $t \in [\tau_{i-1}, \tau_i)$ and depend on the zone number i. It is required to determine the values K_1^i, K_2^i, $i = 1,2,...,L$, optimizing functional (3).

Problem A3. Observation over the state of the process is carried out continuously. According to the results of the observation, the control actions take on the values of zonal controls:

$$u(t) = w^i = const, \quad x(t) \in X^i, \quad t \in [0,T],$$

$$w^i \in U \subset R^r, \ i = 1,2,...,L. \tag{7}$$

It is required to determine the values of zonal control actions w^i, $i = 1,2,...L$, optimizing functional (3).

Note that the problems 1, 2, 3 on the one hand, can be related to parametrical optimal control problems. On the other hand, they can be considered as special problems of finite-dimensional optimization with specifically given target functional.

3. Numerical solution to problem

To solve the posed optimization problems numerically and determine the control actions $u(t)$, $t \in [0,T]$, we propose to use first order optimization methods. As is known, to organize iterative procedure of the first order optimization methods, it is necessary to obtain formulae for the gradient of the target functional [5].

Using the technique of the increment of the target functional obtained at the expense of the increment of one of the optimized arguments, we can prove the following theorems.

Theorem 1. The components of the gradient of the target functional in problem A1 are determined by the following formulae:

$$\frac{\partial J(u)}{\partial v^l} = \int\limits_{X^0} \int\limits_{Y} \frac{\partial I(u; x^0, y)}{\partial v^l} \rho_{X^0}\left(x^0\right) \rho_Y(y)\, dY dX^0,$$

$$\frac{\partial I\left(u;x^0,y\right)}{\partial v^l} = \int\limits_{\Pi_l\left(x^0,y,v\right)} \left[-\psi\left(t;x^0,y,v\right) \cdot \frac{\partial f\left(x\left(t;x^0,y,v\right),v,y\right)}{\partial v} + \frac{\partial g\left(x\left(t;x^0,y,v\right),v\right)}{\partial v}\right] dt, \tag{8}$$

where $\Pi_l\left(x^0,y,v\right) = \bigcup\limits_{j:\,x\left(\tau_j;x^0,y,v\right)\in X^l} \left[\tau_j,\tau_{j+1}\right),\ l = 1,2,...,L;$ $\psi\left(t;x^0,y,v\right)$ the solution to the following conjugate Cauchy problem:

$$\psi\left(T;x^0,y,v\right) = -\Phi_x\left(x\left(T;x^0,y,v\right)\right),$$

$$\dot{\psi}\left(t;x^0,y,v\right) = -\psi^T\left(t;x^0,y,v\right) \cdot$$

$$\cdot\frac{\partial f\left(x\left(t;x^0,y,v\right),v^l,y\right)}{\partial x} - \frac{\partial g\left(x\left(t;x^0,y,v\right),v^l\right)}{\partial x}, \tag{9}$$

$$t \in \Pi_l\left(x^0,y,v\right),\ \ l \in \{1,2,...,L\}$$

under meeting condition (5).

Theorem 2. The components of the gradient of the target functional in problem A2 are determined by the following formulae:

$$\frac{\partial J(u)}{\partial K_1^s} = \int\limits_{X^0}\int\limits_{Y}\int\limits_{\Pi_s\left(x^0,y,K\right)} \left[-\psi^T\left(t;x^0,y,K\right) \cdot \frac{\partial f\left(x\left(t;x^0,y,K\right),K,y\right)}{\partial K} + \frac{\partial g\left(x\left(t;x^0,y,K\right),K\right)}{\partial K}\right] dt \cdot x^T\left(\tau_i;x^0,y,K\right) \cdot \rho_Y(y) \cdot \rho_{X^0}\left(x_0\right) dY dX^0, \tag{10}$$

$$\frac{\partial J(u)}{\partial K_2^s} = \int\limits_{X^0}\int\limits_{Y}\int\limits_{\Pi_s\left(x^0,y,K\right)} \left[-\psi^T\left(t;x^0,y,K\right) \cdot \frac{\partial f\left(x\left(t;x^0,y,K\right),K,y\right)}{\partial K} + \frac{\partial g\left(x\left(t;x^0,y,K\right),K\right)}{\partial K}\right] dt \cdot \rho_Y(y) \cdot \rho_{X^0}\left(x_0\right) dY dX^0, \tag{11}$$

where $\Pi_s\left(x^0,y,K\right) = \bigcup\limits_{j:\,x\left(\tau_j;x^0,y,v\right)\in X^l} \left[\tau_j,\tau_{j+1}\right),$ $s = 1,2,...,L;$ where $\psi\left(t;x^0,y,K\right)$ the solution to the following conjugate Cauchy problem:

$$\psi\left(T;x^0,y,K\right) = -\Phi_x\left(x\left(T;x^0,y,K\right)\right),$$

$$\dot{\psi}\left(t;x^0,y,K\right) = -\psi^T\left(t;x^0,y,K\right) \cdot$$

$$\frac{\partial f\left(x\left(t;x^0,y,K\right),K,y\right)}{\partial x} + \frac{\partial g\left(x\left(t;x^0,y,K\right),K\right)}{\partial x} \tag{12}$$

$$+\sum_{s=1}^{N-1} K_1^s \cdot \delta\left(t-\tau_s\right) \cdot \int\limits_{\tau_s}^{\tau_{s+1}} \left[\frac{\partial g\left(x\left(\tau;x^0,y,K\right),K\right)}{\partial K} - \psi^T\left(\tau;x^0,y,K\right) \frac{\partial f\left(x\left(\tau;x^0,y,K\right),K,y\right)}{\partial u}\right] d\tau,$$
$$t \in [0,T)$$

under meeting condition (6).

Theorem 3. The components of the gradient of the target functional in problem A3 are determined by the following formulae:

$$\frac{\partial J(u)}{\partial w^k} = \int\limits_{X^0}\int\limits_{Y} \frac{\partial I\left(u;x^0,y\right)}{\partial w^k} \rho_{X^0}\left(x^0\right) \rho_Y(y)\, dY dX^0,$$

$$\frac{\partial I\left(u;x^0,y\right)}{\partial w^k} = \int\limits_{\Pi_k\left(x^0,y,w\right)} \left[-\psi\left(t;x^0,y,w\right) \cdot \frac{\partial f\left(x\left(t;x^0,y,w\right),w,y\right)}{\partial w} + \frac{\partial g\left(x\left(t;x^0,y,w\right),w\right)}{\partial w}\right] dt, \tag{13}$$

where $\Pi_k\left(x^0,y,w\right) = \left\{t \in [0,T]:\ \ x\left(t;x^0,y,w\right) \in X^k\right\},$ $k \in \{1,2,...,L\};$ $\psi\left(t;x^0,y,w\right)$ the solution to the following conjugate system:

$$\psi\left(T;x^0,y,w\right) = -\Phi_x\left(x\left(T;x^0,y,w\right)\right),$$

$$\dot{\psi}\left(t;x^0,y,w\right) = -\psi^T\left(t;x^0,y,w\right) \cdot$$

$$\cdot\frac{\partial f\left(x\left(t;x^0,y,w\right),w^k,y\right)}{\partial x} - \frac{\partial g\left(x\left(t;x^0,y,w\right),w^k\right)}{\partial x}, \tag{14}$$

$$t \in \Pi_k\left(x^0,y,w\right),\ \ k \in \{1,2,...,L\},$$

satisfying the following jump condition taking place at the boundary of the zones:

$$\psi\left(t_{l,m}-0;x^0,y,w\right)=\psi\left(t_{l,m}+0;x^0,y,w\right)-$$

$$\frac{\partial}{\partial x}h_{l,m}\left(x\left(t_{l,m};x^0,y,w\right)\right)\cdot\nu_{l,m}, \qquad (15)$$

$$\nu_{l,m}=\frac{\psi\left(t_{l,m}+0;x^0,y,w\right)}{\frac{\partial h_{l,m}\left(x\left(t_{l,m};x^0,y,w\right)\right)}{\partial x}\cdot f\left(x\left(t_{l,m};x^0,y,w\right),w^l,y\right)}\cdot$$
$$\cdot\left[f\left(x\left(t_{l,m};x^0,y,w\right),w^l,y\right)-\right.$$
$$\left.f\left(x\left(t_{l,m};x^0,y,w\right),w^m,y\right)\right].$$

Here $t_{l,m}$, $l,m\in\{1,2,...,L\}$ is the point of time when the trajectory of system (1) hits the boundary of the zones X^l and X^m. The boundary is defined by equation $h_{l,m}(x)=h_{m,l}(x)=0$ with corresponding given functions $h_{l,m}(x)$.

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