Primal and dual multiplicatively barrier methods for linear semidefinite programming problems

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Let $\mathcal{S}^n$ denote the space of real symmetric matrices of order n, and let $\mathcal{S}^n_+$ ($\mathcal{S}^n_{++}$) denote the set of positively semidefinite (positively definite) matrices from $\mathcal{S}^n$. We write also $A \succeq 0$ ($A \succ 0$) to mean that $A \in \mathcal{S}^n_+$ ($A \in \mathcal{S}^n_{++}$).

Consider the linear semidefinite programming problem (SDP)

$$\begin{array}{rcll} \min \; C \bullet X, & & & \\ A_i \bullet X & = & b^i, & i = 1, \ldots, m, \\ X & \succeq & 0, & \end{array} \qquad (1)$$

where all the matrices $C \in \mathcal{S}^n$ and $A_i \in \mathcal{S}^n$, $1 \leq i \leq m$, are given, and $X \in \mathcal{S}^n$ is a variable. $A \bullet B$ denotes the inner product between two matrices A and B of the same size defined by trace $B^T A$.

We also consider the problem dual to (1) in the form

$$\begin{array}{rcl} \max \; b^T u, & & \\ \sum_{i=1}^m u^i A_i + V & = & C, \\ V & \succeq & 0, \end{array} \qquad (2)$$

where $b = (b^1, \ldots, b^m)^T \in \mathbb{R}^m$, $V \in \mathcal{S}^n$.

In what follows we assume that the matrices A_i, $1 \leq i \leq m$, are linear independent. We suppose also that the Slater constraint qualification is fulfilled for both problems (1) and (2), i.e. there are feasible matrices X and V such that $X \succ 0$, $V \succ 0$. In this case the strong duality holds and both problems (1), (2) have nonempty compact sets of solutions [1].

If X_* and V_* are optimal solutions of problems (1) and (2), respectively, then $X_* \bullet V_* = 0$ and the matrices X_* and V_* must commute. Hence, there exists an orthogonal matrix Q such that

$$X_* = Q\mathrm{Diag}(\eta_*)Q^T, \quad V_* = Q\mathrm{Diag}(\theta_*)Q^T,$$

where $\eta_* = [\eta^1_*, \ldots, \eta^n_*]$ and $\theta_* = [\theta^1_*, \ldots, \theta^n_*]$ are the eigenvalues of X_* and V_* respectively. The eigenvalues η^i_* and θ^i_* satisfy the complementarity conditions $\eta^i_* \theta^i_* = 0$, $1 \leq i \leq n$. The *strict complementarity condition* means that, for each $1 \leq i \leq n$, one of the values η^i_* or θ^i_* is strictly positive.

Denote by $X * V$ the symmetrized product of square matrices X and V defined by the formula $X * V = (XV + V^T X^T)/2$. Let $X_0 \succ 0$. Consider the primal iterative method for solving the problem (1)

$$X_{k+1} = X_k - \alpha_k X_k * V_k, \quad V_k = C - \sum_{i=1}^m u^i_k A_i, \quad (3)$$

and u_k is found from the condition

$$A_i \bullet (X_k * V_k) = \tau \left(A_i \bullet X_k - b^i \right), \quad 1 \leq i \leq m, \quad (4)$$

where $\tau > 0$.

Define $\Gamma(X) = \mathcal{A} \bullet (X * \mathcal{A}^T)$, where $\mathcal{A}$ is the $mn \times n$ matrix composed of the matrices A_i, $1 \leq i \leq m$. The (i, j) entry of $\Gamma(X)$ is given by $A_i \bullet (X * A_j)$. Solving the system(4) of linear algebraic equations, we obtain

$$u_k = \Gamma^{-1}(X_k) \left[\mathcal{A} \bullet (X_k * C) + \tau \left(b - \mathcal{A} \bullet X_k \right) \right]. \quad (5)$$

It can be shown that in the case of nondegenerate point X_k the matrix $\Gamma(X_k)$ is nonsingular.

The method (5) can be regarded as a primal interior point method. This method is an extension for (1) of the barrier-projection method, which was previously proposed for solving linear and nonlinear programming problems [2], [3]. For constrained optimization problems depending on vector variables and involving simply structured constraints (for instance, the nonnegativity conditions for the variables), the barrier-projection method makes a transition to other spaces in which such conditions disappear. Then, one can use various constrained optimization methods that work in the entire space. In particular, the gradient projection method can be used for solving the optimization problems with equality constraints. After returning to the original space, the right-hand sides contain additional matrices that play the role of multiplicative barriers and do not allow trajectories to leave the simply structured sets.

For semidefinite programming problems where the variables are matrices, the transition to a new space is replaced by the decomposition of the variable into the product of a nonsingular matrix and its transpose. This decomposition makes it possible to drop the requirement of positive definiteness of matrices. It is important that, similarly to problems with vector variables, one can return to the original matrix variables. The following result is proved in [4].

Theorem 1 *Assume that the strong duality holds for SPP problems (1) and (2) and let their solutions X_* and V_* are strictly complementary. Moreover, let X_* be a nondegenerate vertex of the feasible set in the problem (1). Then for α_k sufficiently small the iterative process (5) locally converges to X_* at a linear rate.*

Using the congruent reduction of two symmetric matrices of which one is positive definite, we obtain $P^T X_k P = I_n$, $P^T(X_k * V_k)P = \mathrm{Diag}(\omega_k)$ for some nonsingular matrix P. Here, the vector $\omega_k = [\omega_k^1, \ldots, \omega_k^n]$ is formed by the eigenvalues of the matrix $Z_k = X_k^{-1} Y_k$. Therefore

$$X_{k+1} = \left(P^{-1}\right)^T \left[I_k - \alpha_k \mathrm{Diag}(\omega_k)\right] P^{-1}.$$

It immediately follows that the matrix X_{k+1} is positive definite if $\alpha_k < 1/\omega_k^{max}$, where ω_k^{max} is the maximal positive eigenvalue.

Now let us consider the variant of the multiplicatively barrier method for solving the dual problem (2). Denote by $\mathrm{vec}A_i$ the direct sum of the columns of A_i. Denote also by $\mathcal{A}_{vec}$ the $m \times n^2$ matrix with $vecA_i$ as its ith row. Applying the decomposition of the matrix V into the product of a nonsingular matrix and its transpose, we obtain the following iterative process

$$\begin{array}{rcl} u_{k+1} & = & u_k + \alpha_k(b - \mathcal{A} \bullet X_k), \\ V_{k+1} & = & V_k - \alpha_k V_k * X_k, \end{array} \tag{6}$$

where $V_0 \succ 0$ and X_k is the symmetric matrix such that

$$\mathrm{vec}X_k = \Gamma^{-1}(V_k)\left[\mathcal{A}_{vec}^T b + \tau(\mathrm{vec}(V_k - C) + \mathcal{A}_{vec}^T u_k)\right],$$

$$\Gamma(V) = \mathcal{A}_{vec}^T \mathcal{A}_{vec} + \mathcal{N}_n (V_k \otimes I_n)\mathcal{N}_n.$$

The matrix $\mathcal{N}_n$ is defined by the formula $\mathcal{N}_n = (I_{n^2} + \mathcal{K}_n)/2$, where $\mathcal{K}_n$ is $n^2 \times n^2$ permutation matrix. For every square matrix M of order n, $\mathcal{K}_n$ effects the permutation $\mathcal{K}_n \mathrm{vec}M = \mathrm{vec}M^T$.

Denote $V(u) = C - \sum_{i=1}^m u^i A_i$. Denote also by $\mathcal{V}_U$ the set of $V(u)$ such that $V(u) \succeq 0$.

Theorem 2 *Assume that the strong duality holds for SPP problems (1) and (2) and let their solutions X_* and V_* are strictly complementary. Moreover, let $V_* = V(u_*)$ be a nondegenerate vertex of the set $\mathcal{V}_U$. Then for α_k sufficiently small the iterative process (5) locally converges to u_* and V_* at a linear rate.*

The other variants of multiplicatively barriers methods for solving primal and dual SDP problems are also considered, and the basic properties of proposed methods are discussed .

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References

[1] E. De Klerk *Aspects of semidefinite programming. Interior point algorithms and selected applications.* Dordrecht etc.: Kluwer Acad. Publs, 2004.

[2] Yu.G. Evtushenko, V.G. Zhadan *Stable Barrier-Projection and Barrier-Newton Methods in Linear Programming.* Comput. Optimaz. and Appl., **3**, 289-304 (1994).

[3] Yu.G. Evtushenko, V.G. Zhadan *Dual Barrier-Projection Methods in Linear Programming.* WSSIAA, **5**, 51-66 (1995).

[4] M.S. Babynin, V.G. Zhadan *A Primal Interior Point Method for Linear Semidefinite Programming Problem.* Computational Mathematics and Mathematical Physics, **48**, 1746-1767 (2008).