New methods for solving nonconvex problems of Optimization and Optimal Control

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1. Introduction

A huge of optimization problems arising from different application areas turn out to be really nonconvex [1]-[6]. The most of such problems deals with (d.c.) functions which can be represented as a difference of two convex functions. Besides this class (DC) possesses several remarkable properties which make it the most attracting object in modern optimization [2, 4].

a) $DC(I\!R^n)$ is generated by mostly investigated class — the cone of convex functions — and turns out to be the vector space.

b) $DC(I\!R^n)$ contains the spaces $C^2(I\!R^n)$ and power and trigonometrical polynoms and so on.

c) Any continuous function over a compact on $I\!R^n$ can be approximated by some d.c. function with an arbitrary accuracy, so that any nonlinear (continuous) optimization problem on a compact can be viewed as a d.c. optimization problem with some accuracy.

The situation in Nonconvex (Global) Optimization may be viewed, at present, as dominated by B&B approach [2, 5, 6]. On the other side the classical methods of convex optimization have been thrown aside because of its inefficiency [1, 2, 7]. As well-known the conspicuous limitation of convex optimization methods applied to nonconvex problems is their ability of being trapped at a local extremum or even a critical point depending on a starting point [1, 2]. So, the classical approach shows itself inoperative for new problems arising from practice. In such a situation we advanced another approach [2, 3] the core of which is composed by Global Optimality Conditions (GOC) for principal classes of d.c. programming problems.

2. Local Search Methods (LCM)

In difference of well-known B&B, cuts and similar methods throwing away classical methods of convex optimization, we insist on certain, but nondirect, application of these algorithms in Global Optimization. For example, in d.c. minimization problem

$$(\mathcal{P}): \qquad f(x) = g(x) - h(x) \downarrow \min_x, \; x \in D, \qquad (1)$$

where g, h, D are convex, the following (partially) linearized problem is the basic element (a "brick")

$$(\mathcal{PL}_s): \; f(x) = g(x) - \langle \nabla h(x^s), x\rangle \downarrow \min_x, \; x \in D, \; (2)$$

where x^s is a current iterate. It means that the choice of a solving method for $(\mathcal{PL}_s)$ has a considerable impact on Global Search. Local Search procedure for $(\mathcal{P})$ may consists in consecutive solving the $(\mathcal{PL}_s)$: knowing $x^s \in D$, we find $x^{s+1} \in D$ as an approximate solution of $(\mathcal{PL}_s)$–(2). Unexpectedly, the process tends to a solution x_* of the linearized problem

$$(\mathcal{PL}_*): \; f(x) = g(x) - \langle \nabla h(x_*), x\rangle \downarrow \min_x, \; x \in D. \; (3)$$

So, the point $x_* \in D$ turns out to be critical.

3. Global Search procedures

The general procedure of Global Search consists of two stages:

a) Local Search;

b) Procedure of escape from a critical point based upon GOC.

The meaning of this combination consists in the algorithmic (constructive) property of GOC providing a better feasible point when GOC are broken down. Actually, for $(\mathcal{P})$–(1) GOC are, as follows,

$$z \in Sol(\mathcal{P}) \Rightarrow \forall (y,\beta) \in I\!R^n \times I\!R:$$

$$\left.\begin{array}{l} h(y) = \beta - \xi, \;\; \xi := g(z) - h(z) \stackrel{\triangle}{=} f(z) \\ g(x) - \beta \geq \langle \nabla h(y), x - y\rangle \;\; \forall x \in D. \end{array}\right\} \qquad (4)$$

If some $(\hat{y}, \hat{\beta})$ in (4) and $\hat{x} \in D$ are such that $g(\hat{x}) < \beta + \langle \nabla h(\hat{y}), \hat{x} - \hat{y}\rangle$, then due to convexity of $h(\cdot)$ we have

$$f(\hat{x}) = g(\hat{x}) - h(\hat{x}) < h(\hat{y}) + \xi - h(\hat{y}) = f(z)$$

so that $f(\hat{x}) < f(z)$, and $\hat{x}$ is better than z.

By varying the parameter $(y, \beta) \in I\!R^n \times I\!R$ in (4), and by solving the corresponding linearized problems (sf(4))

$$g(x) - \langle \nabla h(y), x \rangle \downarrow \min_{x},\ x \in D, \tag{5}$$

(y is not obligatory feasible!) we get a family of starting points $x(y, \beta)$ for LSM. Besides, no needs to consider all (y, β), it is sufficient to violate the inequality in (4) for one pair $(\hat{y}, \hat{\beta})$ only.

The large field of computational experiments confirmed the effectiveness of the approach for high dimensional problems even in the case of programm implementation performed by students and post-graduation students [8]-[13].

4. Application Problems

4.1. Bimatrix games (BMG). A new method of finding the Nash equilibrium in BMG has been developed [3]. This one is based on reducing BMG to a bilinear maximization problem and a following application of Global Search Strategy. The testing on widely generated BMG of high dimension (up to 1000×1000) showed the comparable effectiveness of the method.

4.2. Bilevel problems can reviewed as extremal problems having a special constraint in the form of another optimization problem (follower problem). Besides, the usual constraints are depending on the variables of the follower. A special complex of programmes for solving these problems was developed and successfully tested on a large number stochastically generated examples of different complexity and dimension (up to 150×150).

4.3. Linear complimentarity problem was solved by variational approach stating it as d.c. minimization problem of dimension up to 400 [13].

4.4. Problems of financial and medical diagnostic can be formulated in the form of nonlinear (polyhedral) separability. The generalization of Global Search Theory for nonsmooth case allows to develop a programming complex for solving such problems of rather high dimension with demonstrated its effectiveness during computational simulations.

4.5. Well-known problems of **Discrete programming** [8] and nonconvex problems of **Optimal control** [11, 12] have been also considered.

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