Approximate Solution of Auxiliary Problems in the Level Method

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The level method was developed in [1], [2] as an efficient oracle-type algorithm to solve optimization problems and to find saddle points. It was extended further in [3]–[7] making use of more complicated oracle schemes, leading to generalized level methods.

The layout of the level method is as follows.

Let E be a finite-dimensional space and $Q \subset E$ be a polyhedron containing a nonempty convex compact set Q^*. We want to find a point of Q^* or an approximation of such a point within a certain tolerance. A nonempty set of responses of the oracle $\mathbf{B}(z) = \{(b(z), \beta(z)\}$ is assigned to each point $z \in Q$. An oracle's response is a pair $(b(z), \beta(z))$, where $b(z)$ is a vector and $\beta(z)$ is a scalar. The set $\mathbf{B}(z)$ contains the pair $b(z) = 0$, $\beta(z) = 0$ only for the points $z \in Q^*$. In other pairs, $\|b(z)\| \neq 0$, and every half-space composed of the points $y \in E$ satisfying the inequality

$$\langle b(z), y - z\rangle + \beta(z) \le 0, \tag{1}$$

contains the set Q^*.

In the level method, for each $z \in Q \setminus Q^*$, the oracle produces one or several pairs formed of vectors $b = b(z)$ ($\|b(z)\| \neq 0$) and nonnegative scalars $\beta = \beta(z)$, $(b, \beta) \in \mathbf{B}(z)$, such that each pair satisfies (1); if the oracle finds that $b(\overline{z}) = 0$ and $\beta(\overline{z}) = 0$ for a point $\overline{z} \in Q$, then $\overline{z} \in Q^*$.

Let us describe a variant of the generalized level method that uses the above-mentioned oracle. Define the sets $S_0 = \varnothing$, $\Lambda_0 = \varnothing$, $Q_0 = Q$, and the number $n_0 = 0$. Select a symmetric positive definite matrix H and an arbitrary point $z_1 \in Q$.

Assume that k ($k \ge 1$) points $z_i \in Q$ ($i = 1, 2, \dots, k$) have already been generated, and the set S_{k-1}, $|S_{k-1}| = n_{k-1}$ has been defined. With the point z_k, we associate m_k ($m_k \ge 1$) oracle's responses, i.e., we define m_k pairs containing vectors $b_j = b_j(z_k)$ and scalars $\beta_{j,k} = \beta_{j,k}(z_k)$ ($j \in J_k = \{n_{k-1}+1, \dots, n_{k-1}+m_k\}$) such that each pair satisfies condition (1). Set $S_k = S_{k-1} \cup J_k$, where $|J_k| = m_k$ and $|S_k| = n_k = n_{k-1} + m_k$. Furthermore, we assume that $\|b_j\| \neq 0$ for all $j \in J_k$ (otherwise, we have $z_k \in Q^*$). Besides, we suppose that, for each $k \ge 1$, there exists an index $\widehat{\jmath} \in J_k$ such that $\beta_{\widehat{\jmath},k} \ge 0$.

We also assume that the polyhedron Q_k defined by the requirements

$$\{z \in Q : \ \langle b_j, z - z_i\rangle + \beta_{j,k} \le 0, \ \ j \in J_i, \ \ i = 1, \dots, k\},$$

where $\beta_{j,k} \ge \beta_{j,k-1}$ for all $j \in J_i$ ($i = 1, \dots, k-1$), comprises Q^* (hence is nonempty).

As usual, the search of the next point z_{k+1} in the level method starts by solving the linear program $\mathcal{A}_k$

$$\begin{aligned} &t \to \max, \quad z \in Q, \\ &\langle b_j, z - z_i\rangle + \beta_{j,k} + t \le 0, \quad j \in J_i, \quad i = 1, \dots, k, \end{aligned} \tag{2}$$

with the variables z and t. Denote its optimal value by $\Delta_k \ge 0$. Due to their definition, the numbers Δ_k do not increase by k.

If Δ_k is small enough, the process is stopped. Otherwise, we add $m_k \ge 1$ numbers $\lambda_j \in (0, 1)$ ($j \in J_k$) to the set Λ_{k-1}; i.e., we determine $\Lambda_k = \Lambda_{k-1} \cup \{\lambda_j, j \in J_k\}$; and the (nonempty) polyhedron $Q_k(\Lambda_k, \Delta_k)$ is defined by the inequalities

$$\begin{aligned} \{z \in Q : \quad &\langle b_j, z - z_i\rangle + \beta_{j,k} + \lambda_j \Delta_k \le 0, \\ &j \in J_i, \quad i = 1, \dots, k\}. \end{aligned} \tag{3}$$

To complete iteration k, a new point z_{k+1} is found as the projection of z_k onto the polyhedron $Q_k(\Lambda_k, \Delta_k)$; i.e., the quadratic programming problem $\mathcal{B}_k$

$$\langle H(z - z_k), z - z_k\rangle \to \min, \quad z \in Q_k(\Lambda_k, \Delta_k), \tag{4}$$

is solved.

In what follows, we assume that $0 < \underline{\lambda} \le \lambda_j \le \overline{\lambda} < 1$ for all $\lambda_j \in \Lambda_k$, and $\|b_j\| \le \overline{B}$ for each $j \in S_k$ ($k \ge 1$), where the oracle's responses are defined by the system of inequalities (1). Suppose that d is the diameter of Q, and $\mu = h_{\max}/h_{min}$ is the condition number of the matrix H.

Lemma 1 *The sequence $\{\Delta_k\}$ generated by the above-described level method, satisfies the inequality:*

$$\Delta_k \le \frac{\overline{B} d \sqrt{\mu}}{\underline{\lambda}\sqrt{1 - \overline{\lambda}^2}}\, k^{-1/2}, \qquad k = 1, 2, \dots. \tag{5}$$

Lemma 1 is proved [6] under assumptions that when determining the set $Q_k(\Lambda_k, \widehat{\Delta}_k)$ and hence, a new point z_{k+1}, it is allowed to make use of an inexact (approximate) solution $\widehat{\Delta}_k$ (with $c\Delta_k \le \widehat{\Delta}_k \le \Delta_k$, $0 < c \le 1$) of the problem $\mathcal{A}_k$, in place of its exact solution Δ_k. Moreover, the quadratic problem $\mathcal{B}_k$ may be also solved inexactly.

Let $z_{k+1} \in G$ be an approximate solution to the quadratic program $\mathcal{B}_k$, that is, a point in $Q_k(\Lambda_k, \Delta_k)$ such that

$$\langle H(z_k - \overline{z}_{k+1}), z_k - \overline{z}_{k+1}\rangle \le \langle H(z_k - z_{k+1}), z_k - z_{k+1}\rangle \le \langle H(z_k - \overline{z}_{k+1}), z_k - \overline{z}_{k+1}\rangle(1+\tau_k^2), \qquad (6)$$

where τ_k^2 $(k = 1, 2, \ldots)$ is the relative error in the approximate solution z_{k+1} of the quadratic problem $\mathcal{B}_k$, with respect to its exact solution $\overline{z}_{k+1}$.

Applying the decomposition $H = \overline{H}^T\overline{H}$, we can obtain the following inequality

$$\|\overline{H}(z_{k+1} - \overline{z}_{k+1})\| \le \tau_k \|\overline{H}(z_k - z_{k+1})\|,$$

whence τ_k can be regarded as the relative error (with respect to the metric $\|\cdot\|_H$) in an approximate solution of the quadratic problem $\mathcal{B}_k$.

Lemma 2 *With all the assumptions of Lemma* 1 *valid, suppose that problem $\mathcal{B}_k$ is solved approximately with the relative error τ_k $(k \ge 1)$. Also, assume that conditions* (6) *hold, and let*

$$S = \sum_{k=1}^{\infty} (2\tau_k + \tau_k^2), \qquad (7)$$

where

$$\sum_{k=1}^{\infty} \tau_k < \infty. \qquad (8)$$

Then the sequence $\{\Delta_k\}$ of (exact or approximate) solutions to the problems $\mathcal{A}_k$ generated by the level method satisfies condition (5) *with $\overline{B}(1+S)$ substituted for $\overline{B}$.*

Lemma 3 *Replace condition* (8) *in Lemma* 2 *by the requirements*

$$0 \le \tau_k \le \mathbf{C}\|\overline{H}(z_k - z_{k+1})\|, \quad 0 \le \mathbf{C} < \overline{\mathbf{C}} \equiv \frac{\sqrt{2}-1}{d\sqrt{h_{\max}}}. \qquad (9)$$

Then the sequence $\{\Delta_k\}$ of (exact or approximate) solutions to the problems $\mathcal{A}_k$ generated by the level method satisfies condition (5) *with $\overline{B}(1 - \widetilde{S})^{-1}$ substituted for $\overline{B}$, where*

$$\widetilde{S} = 2\mathbf{C}d\sqrt{h_{\max}} + \mathbf{C}^2 d^2 h_{\max}, \qquad 0 \le \widetilde{S} < 1. \qquad (10)$$

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