OPTIMAL CONTROL PROBLEM FOR PARABOLIC EQUATIONS WITH NONSMOOTH NONLINEARITY AND FIXED FINAL STATE

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We have an optimal control problem for the parabolic equation with a nonsmooth nonlinear term and a fixed terminal state. Let Ω is n-dimensional bounded set, $T > 0$, $Q = \Omega \times (0, T)$. We consider the parabolic equation

$$y' - \Delta y + a(s, y) = v$$

in the set Q with homogeneous Dirichlet boundary conditions and homogeneous initial condition, where $s = (x, t)$. The function a belongs to the class A of monotonous Caratheodory functions satisfying to the conditions

$$|a(s, \varphi)| \leq a_0(s) + b|\varphi|^{q-1}, \; a(s, \varphi) \geq c|\varphi|^q$$

for all $s \in Q$, $\varphi \in \mathrm{R}$, where $a_0 \in L_{q'}(Q)$, $b > 0$, $c > 0$, $q > 1$, $1/q + 1/q' = 1$. This boundary problem has a unique solution $y = y[v]$ from the set

$$Y = \{y | \; y \in X, \, y' \in X'\}$$

for all $v \in X'$, where $X = Z \cap L_q(Q)$, X' is the conjugate space of X, $Z = L_2(0, T; H_0^1(\Omega))$.

We have also the functional

$$I(v) \; = \int\limits_Q f(v, y[v], \nabla y[v]) dQ \; + \; \chi \int\limits_Q v^2 dQ,$$

where $\chi > 0$, f belongs to the class F of Caratheodory functions, satisfying to the following properties: $f(s, \varphi) \geq \alpha|\varphi|^2$ for all $s \in Q$, $\varphi \in \mathrm{R}^{n+1}$ with $\alpha > 0$; besides $f(s, \psi, \cdot\,)$ is convex for all $s \in Q$, $\psi \in \mathrm{R}$. We would like to minimize the functional I on the subset U_∂ of the nonempty convex closed set U from X', which guarantees the final condition

$$y[v]|_{t=T} = z,$$

where z is a given function from the space $L_2(\Omega)$. This problem is solvable.

Our problem has three difficulties: the fixed terminal state, the absence of the smoothness of the functions a, f and the absence of the restriction on the parameter of nonlinearity q. We shall overcome the first difficulty by means of the smooth approximation[1], the second one with using of the penalty method[2], and the third one by means of the extanded differentiation[3].

We consider consequences $\{a_k\}$ of the class A and $\{f_k\}$ of the class F with inclusions $a_k(s, \cdot\,) \in C^1(\mathrm{R})$, $f_k(s, \cdot\,) \in C^1(\mathrm{R}^{n+1})$ for all $s \in Q$, $k = 1, 2, ...$, with uniformly convergence $a_k(s, \varphi) \to a(s, \varphi)$ and condition

$$\lim_{k\to\infty} \sup_{y \in Y} \int\limits_Q |f_k(s, y, \nabla y) - f(s, y, \nabla y)| dQ \; = \; 0.$$

Let us consider the homogeneous Dirichlet boundary problem for the equation

$$y_k' - \Delta y_k + a_k(s, y_k) = v.$$

It has a unique solution $y_k = y_k[v]$ from Y for $v \in X'$. We determine also the functional

$$I_k(v) \; = \int\limits_Q f_k(v, y_k[v], \nabla y_k[v]) dQ \; + \; \chi \int\limits_Q v^2 dQ \, +$$

$$+ \, (\varepsilon_k)^{-1} \| y_k[v]|_{t=T} - z \|^2_{H^{-1}(\Omega)},$$

where $\varepsilon_k > 0$ and $\varepsilon_k \to 0$ if $k \to \infty$. The approximational problem is the minimization of this functional on the set U. It has a solution v_k.

Theorem 0.1 *We can choose the number k so large, that the inclusion $y[v_k]|_{t=T} \in z + O$ and the inequality $I(v_k) \leq \inf I(U_\partial) + \delta$ are true for the arbitrary small weak neighbourhood O of zero in the space $L_2(\Omega)$ and the arbitrary value $\delta > 0$.*

The solution of the approximational problem for the large enough k could be used as the approximate solution of the initial problem, because the final state constraint is true with a high enough exactness, and the corresponding value of the state functional I is not far from its minimum on the admissible control set. Therefore we reduce the initial problem to the finding of the solution of the approximational optimal control problem with the smooth parameters and the free terminal state.

The classical necessary condition of the minimum of the Gataux differentiable functional J on the convex set U in the point u is the variational inequality

$$\langle J'(u), v-u\rangle \ \geq 0 \ \forall v \in U,$$

where $J'(u)$ is the derivative of J in the point u, $\langle \lambda, \varphi\rangle$ is the value of the linear continuous functional λ in the point φ. Our approximational functional depends from the approximational state $y_k[v]$. So we need to find the derivative of the control-state-mapping $y_k[\cdot\,] : X' \to Y$ for the getting of the differentiability of the functional I_k. Unfortunately we have the following result.

Theorem 0.2 *The mapping $y_k[\cdot\,] \, : \, X' \, \to \, Y$ is not Gataux differentiable for enough large value of the dimension n of the set Ω and the parameter q of nonlinearity.*

This result prohibits from the differentiation of the functional by means of the classical method. However we have more weak property. Let v is an arbitrary point from X'. We determine the space

$$Y(v) \ = \ \{p \in Z | \ | \int\limits_Q a_{ky}(s, y[v])p^2 dQ| < \infty\}$$

and its conjugate space $X(v)$, where $a_{ky}(s,\varphi)$ is the derivative of the function $a_{ky}(s,\cdot\,)$ in the point φ.

Theorem 0.3 *There is the convergence*

$$\frac{y_k[v + \ \sigma h] - y_k[v]}{\sigma} \to D(v)h \text{ in } Z$$

for all $h \in Z'$ if $\ \sigma \to 0$, where the linear continuous operator $D(v) : X(v) \to Y(v)$ satisfies to the equality

$$\int\limits_Q \mu D(v)hdQ \ = \ \int\limits_Q p_\mu(v)hdQ \ \forall \mu, v \in X(v),$$

where $p_\mu(v)$ is the solution of the equation

$$-p'_\mu(v) - \Delta p_\mu(v) + a_{ky}(s, y[v])p_\mu(v) \ = \ \mu$$

with homogenous Dirichlet boundary condition and homogenous terminal value.

Now we can find the derivative of the functional I_k.

Theorem 0.4 *The functional I_k has the derivative $I'_k(v_k) \, = \, \chi v_k + p_k$ with respect to the subspace Z' in the point v_k, where p_k is the solution of the equation*

$$-p'_k - \Delta p_k + a_{ky}(s, y[v_k])p_k \ =$$

$$= \ f_{ky}(s, y[v_k], \nabla y[v_k]) - \mathrm{div} f_{k\nabla y}(s, y[v_k], \nabla y[v_k])$$

with homogeneous boundary condition and final condition

$$p_k|_{t=T} \ = \ \Lambda(y(v_k)|_{t=T} - z)/\varepsilon_k,$$

Λ is the canonical isomorphism from $H_0^1(\Omega)$ to $H^{-1}(\Omega)$.

Then we get the necessary condition of optimality for the approximational problem.

Theorem 0.5 *The solution v_k of the approximational optimal control problem satisfies to the variational inequality*

$$\int\limits_Q (\chi v_k + p_k)(v - v_k)dQ \ \geq \ 0 \ \forall v \in U.$$

Thus we could find the solution of the approximational problem from the system, including the considered boundary problem, conjugate boundary problem and the last variational inequality. It will be the approximate solution of the initial optimal control problem for enough large number k.

We consider as an addition the controllability of our system. Let us name this system controllable if the set

$$\{y[v]|_{t=T} | \ v \in X'\}$$

is dense in the space $L_2(\Omega)$. Thus it is controllable if for all $z \in L_2(\Omega)$ and $\delta > 0$ we can choose a control $v \in X'$ such as

$$\|y[v]_{t=T} - z\|^2_{L_2(\Omega)} \leq \delta.$$

Theorem 0.6 *Let's $n \, \leq \, 2$ or $q \, \leq \, 2n/(2n-2)$ for $n > 2$. Then the system is controllable if the inequality $|a(s,\varphi)| \leq a_0(s) + b|\varphi|^{q/2}$ for all $s \in Q$, $\varphi \in \mathrm{R}$ is true with $a_0 \in \ L_2(Q)$, $b > 0$.*

References

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