Numerical search for optimistic solutions in nonlinear bilevel problems

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During the recent decades, problems with hierarchical (in particular, bilevel) structure seem to be the most attractive field for many experts. In course of investigation of bilevel programming problems the difficulty arises already at the stage of defining a concept of solution. The optimistic and pessimistic (guaranteed) concepts of solution are the most popular. [1]–[4].

During the three decades of intensive investigation of bilevel programming problems there were many methods of finding the optimistic solutions proposed by different authors (see the surveys [3], [4]). Nevertheless, as far as we can conclude on the basis of available literature, there are only a few published results containing numerical solutions of even test bilevel high-dimension problems (e.g. problems with the dimension up to 200). So, development of new numerical methods for nonlinear bilevel problems, while implying verification of their efficiency by numerical testing, is one of the most important problems of operations research.

This work is devoted to elaboration of new techniques for finding optimistic solutions of bilevel problems, where the upper level goal function is d.c. (represented by difference of two convex functions), and the lower level goal function is quadratic. So that the problem is formulated as follows:

$$\left.\begin{array}{c} F(x,y) \stackrel{\triangle}{=} g(x,y) - f(x,y) \downarrow \min\limits_{x,y}, \\ (x,y) \in X \stackrel{\triangle}{=} \{(x,y) \in I\!R^{m+n} \mid \\ \mid h_l(x,y) \le 0,\ l = 1,...,p\}, \\ y \in Y_*(x) \stackrel{\triangle}{=} \operatorname*{Argmin}\limits_y \{\frac{1}{2}\langle y, Cy\rangle + \\ +\langle xQ + d, y\rangle \mid Ax + By \le b\}, \end{array}\right\} \qquad (\mathcal{P})$$

where the functions $g(\cdot)$, $f(\cdot)$, $h_l(\cdot)$, $l = 1,...,p$ are convex on $I\!R^{m+n}$, $d \in I\!R^n$, $b \in I\!R^q$, A, B, C, Q are matrices of appropriate dimension, and $C = C^T$ is nonnegatively defined.

Also we investigate a particular case of problem $(\mathcal{P})$ with quadratic goal function on the upper and lower levels:

$$\left.\begin{array}{c} \frac{1}{2}\langle x, Cx\rangle + \langle c, x\rangle + \frac{1}{2}\langle y, Dy\rangle + \langle d, y\rangle \downarrow \min\limits_{x,y}, \\ (x,y) \in X \stackrel{\triangle}{=} \{(x,y) \in I\!R^{m+n} \mid Ax + By \le b\}, \\ y \in Y_*(x) \stackrel{\triangle}{=} \operatorname*{Argmin}\limits_y \{\frac{1}{2}\langle y, D_1 y\rangle + \\ +\langle xQ_1 + d_1 y\rangle \mid A_1 x + B_1 y \le b_1\}, \end{array}\right\} \qquad (\mathcal{P}_1)$$

where $c \in I\!R^m$; $d, d_1 \in I\!R^n$; $b \in I\!R^p$; $b_1 \in I\!R^q$; A, B, C, D, Q, A_1, B_1, D_1, Q_1 — are matrices of appropriate dimension, $C = C^T \ge 0$, $D = D^T \ge 0$, $D_1 = D_1^T \ge 0$.

Such bilevel problems may be reduced to one or several single-level nonconvex (d.c.) problems via, for instance, the KKT-rule (see, for example, [1], [2]):

$$\left.\begin{array}{c} g(x,y) - f(x,y) + \mu\langle v, b - Ax - By\rangle \downarrow \min\limits_{x,y,v}, \\ (x,y,v) \in D \stackrel{\triangle}{=} \{(x,y,v) \mid (x,y) \in X,\ v \ge 0, \\ Cy + d + xQ + vB = 0,\ Ax + By \le b\}; \end{array}\right\} \qquad (\mathcal{DC})$$

$$\left.\begin{array}{c} \frac{1}{2}\langle x, Cx\rangle + \langle c, x\rangle + \frac{1}{2}\langle y, Dy\rangle + \langle d, y\rangle + \\ +\mu\langle v, b_1 - A_1 x - B_1 y\rangle \downarrow \min\limits_{x,y,v}, \\ (x,y,v) \in D \stackrel{\triangle}{=} \{(x,y,v) \mid Ax + By \le b, \quad v \ge 0, \\ A_1 x + B_1 y \le b_1, \quad D_1 y + d_1 + xQ_1 + vB_1 = 0\}, \end{array}\right\} \qquad (\mathcal{DC}_1)$$

where $\mu > 0$ is a penalty parameter,

It is known, that nonconvex problems may have a large number of local solutions, which are far – even from the viewpoint of the goal function's value – from a global one [5], [6].

Direct application of standard convex optimization methods [5] turns out to be inefficient from the view point of global search. So, there appears the need to construct new global search methods, allowing to escape from a stationary (critical) point.

For the purpose of solving the problems formulated above, we intend to construct the algorithms based on the Global Search Theory (GST) in d.c. programming problems developed in [6]–[11]. Global Search Algorithms based on GST consist of two principal stages: 1) a special local search methods, which takes into account the structure of the problem under scrutiny; 2) the procedures, based on Global Optimality Conditions, which allow to improve the point provided by the Local Search Method [6]–[11].

In particular, a Local Search in problem $(\mathcal{DC})$ consists in the consecutive (approximate) solving the convex linearized problems of the form $((x^s, y^s, v^s) \in D)$

$$\left.\begin{array}{c} g(x,y) + \dfrac{\mu}{4}(4\langle v, b\rangle + \|v - Ax\|^2 + \|v - By\|^2) - \\ -\langle \nabla_{xy} f(x^s, y^s)(x,y)\rangle - \dfrac{\mu}{2}(\langle v^s + Ax^s, v\rangle + \\ +\langle (v^s + Ax^s)A, x\rangle + \langle v^s + By^s, v\rangle + \\ +\langle (v^s + By^s)B, y\rangle) \downarrow \min\limits_{x,y,v}, \quad (x,y,v) \in D. \end{array}\right\} \tag{$\mathcal{PL}$}$$

Linearization in the problem $(\mathcal{PL})$ is performed for the basic (generic) nonconvexity of a problem $(\mathcal{DC})$, and problem $(\mathcal{PL})$ can be solved by standard software packages.

To the end of a local search for Problem $(\mathcal{DC}_1)$ we apply the idea of consecutive solving partial problems with respect to two groups of variables (see [8]–[11]). In order to do it, we separate the pair (x, y) and the variable v. For a fixed value of variable v problem $(\mathcal{DC}_1)$ becomes a convex quadratic optimization problem, and for a fixed pair (x, y) we obtain a problem of linear programming with respect to v $((x^s, y^s, v^s) \in D)$:

$$\left.\begin{array}{c} \dfrac{1}{2}\langle x, Cx\rangle + \langle c, x\rangle + \dfrac{1}{2}\langle y, Dy\rangle + \langle d, y\rangle - \\ -\mu(\langle v^s A_1, x\rangle + \langle v^s B_1, y\rangle) \downarrow \min\limits_{x,y}, \\ Ax + By \le b, \quad A_1 x + B_1 y \le b_1, \\ D_1 y + d_1 + x Q_1 + v^s B_1 = 0, \end{array}\right\} \tag{$\mathcal{QP}$}$$

$$\left.\begin{array}{c} \langle b_1 - A_1 x^s - B_1 y^s, v\rangle \downarrow \min\limits_{v}, \\ v \ge 0, \quad D_1 y^s + d_1 + x^s Q_1 + v B_1 = 0. \end{array}\right\} \tag{$\mathcal{LP}$}$$

These auxiliary problems can be solved with the help of standard software packages also.

The procedures of Global Search for problems $(\mathcal{DC})$ and $(\mathcal{DC}_1)$ based on the corresponding strategy of global search for problems of d.c. minimization [6]–[11] because the goal function in problems of such kind may be represented as a difference of two convex functions. In combination with directed selection, in the process of increasing the the value of parameters $\mu > 0$, the procedures of global search forms a methods for solving problems $(\mathcal{P})$ and $(\mathcal{P}_1)$. The crucial moment of Global Search procedures consists in constructing an approximation of the level surface of the convex function, which generates the basic nonconvexity in the problem under consideration. For the purpose of constructing such an approximation we have to take account of the information related to the problems statements.

Computational testing of the elaborated methods has shown the efficiency of the proposed approach.

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