Solving strongly monotone variational and quasi-variational inequalities

Yu. Nesterov*, L. Scrimali†

*CORE, Université Catholique de Louvain, Yurii.Nesterov@uclouvain.be
†University of Catania, Scrimali@dmi.unict.it

Let $Q \subseteq E$ be a closed convex set. Consider a continuous operator $g(x) : \ Q \to E^*$, which is *strongly monotone* with constant $\mu > 0$. The *variational inequality* problem (VI) consists in finding $x^*(Q) \in Q$:

$$\langle g(x^*(Q)), y - x^*(Q) \rangle \geq 0 \quad \forall y \in Q. \tag{1}$$

In order to describe the quality of approximate solutions to (1), we introduce the following *merit function*:

$$f(x) = \sup_{y \in Q} \left\{ \langle g(y), x - y \rangle + \tfrac{1}{2}\mu \|y - x\|^2 \right\}. \tag{2}$$

where the norm $\| \cdot \|$ is Euclidean.

Theorem 1 *Merit function $f(x)$ is well defined and strongly convex on Q with convexity parameter μ. Moreover, it is non-negative on Q and vanishes only at the unique solution of variational inequality (1).*

For $\beta > 0$, denote

$$\begin{array}{rcl} \psi_y^\beta(x) & = & \langle g(y), y - x \rangle - \frac{1}{2}\beta \|x - y\|^2, \\ \Psi_k(x) & = & \sum\limits_{i=0}^{k} \lambda_i \psi_{y_i}^\mu(x). \end{array}$$

Let g be Lipschitz continuous on Q with constant L, and $\bar{x} \in Q$. Denote $\gamma = \frac{L}{\mu} \geq 1$. Consider the method

$$\begin{array}{ll} \textbf{Input}: & \text{Set } \lambda_0 = 1, \text{ and } y_0 = \bar{x}. \\ \textbf{Iteration} & (k \geq 0)\textbf{:} \\ & x_k = \arg\max\limits_{x \in Q} \Psi_k(x), \\ & y_{k+1} = \arg\max\limits_{x \in Q} \psi_{x_k}^L(x), \\ & \lambda_{k+1} = \frac{1}{\gamma} \cdot S_k. \\ \textbf{Output}: & \tilde{y}_k = \frac{1}{S_k} \sum\limits_{i=0}^{k} \lambda_i y_i. \end{array} \tag{3}$$

Theorem 2 *For any $k \geq 0$, we have*

$$\tfrac{\mu}{2} \cdot \|\tilde{y}_k - x^*\|^2 \ \leq \ f(\tilde{y}_k) \ \leq \ f(\bar{x}) \cdot \gamma^2 \cdot e^{-k/(\gamma+1)}.$$

Note that the this process is much faster than the *gradient method* which converges as $O(e^{-k/\gamma^2})$.

Consider now *quasi-variational inequalities* [1]. Let $\mathcal{Q} : E \to 2^E$ be a multifunction with nonempty closed and convex values. We are interested in the following problem (QVI): find $x_* \in \mathcal{Q}(x_*)$ such that

$$\langle g(x_*), y - x_* \rangle \geq 0, \quad \forall y \in \mathcal{Q}(x_*). \tag{4}$$

In order to prove the existence of its solution, we need to assume that the set $Q(x)$ is not changing too quickly.

Let us introduce the *relaxation operator* $T(x) = x^*(\mathcal{Q}(x))$ defined by the following relations:

$$\begin{array}{rcl} T(x) & \in & \mathcal{Q}(x), \\ \langle g(T(x)), y - T(x) \rangle & \geq & 0 \quad \forall y \in \mathcal{Q}(x). \end{array} \tag{5}$$

Clearly, the solution to (4) is a fixed point of $T(x)$.

Theorem 3 *Assume that there exists some $\alpha \geq 0$ such that for all $x, y, z \in E$ we have*

$$\|proj_{\mathcal{Q}(x)}(z) - proj_{\mathcal{Q}(y)}(z)\| \quad \leq \quad \alpha \|x - y\|.$$

Then $T(x)$ is Lipschitz continuous with constant $\alpha\gamma$.

Corollary 1 *If $\alpha < \gamma^{-1}$, then there exists a unique solution to problem (4).*

It appears that the most efficient way for solving QVI is an *approximate tracing* of the fixed-point iterates $x_{k+1} \approx T(x_k)$ by the method (3). It is much faster then the gradient-type technique [2].

References

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[2] M. Kocvara M., J.V. Outrata. On a class of quasi-variational inequalities. *Optimization Methods and Software*, **5**, 275-295 (1995).